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An application of Ψ-spaces

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Abstract

Ψ-spaces were first introduced by J. R. Isbell and were also discussed by L. Gillman and M. Jerison in their famous book Rings of Continuous Functions$[3]$ $[5]$. S. P. Franklin used Ψ-space to give an example of a compact, Hausdorff, sequential, non Fréchet-Urysohn space$[4]$. Here we make use of the Ψ-space to construct a space which is not homeomorphic to the sequential fan (in fact not homeomorphic to any one of the six examples$[1]$), though it is a countable, Fréchet-Urysohn space with unique limit point.

Keywords: First countable, Fréchet-Urysohn, Sequential fan, pairwise almost disjoint family, Ψ-space.

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Throughout, N denotes the set of natural numbers and Q denotes the set of rational numbers.

Fréchet-Urysohn:

A topological space X is called a Fréchet-Urysohn space if for every subset A of X and every $x \in \overline{A}$, there exists a sequence (x_n) of points of A converging to x.

Note that every first-countable space is Fréchet-Urysohn; there are many examples showing that converse is not true. One such example is known as the sequential fan.

Sequential fan:

Consider countably many disjoint copies of a convergent sequence (i.e. copies of $\left\{\frac{1}{n} / n \in N\right\} \cup \{0\}$ as subsets of the real line) and identify the limit points, denote this new identified point by 0 and the resulting space by the set F. New space F is called sequential fan$[1]$.

The sequential fan has following properties:

1. Countable
2. Not first countable
3. Fréchet-Urysohn

Diagonal Sequence Condition:

Peter J. Nyikos stated the following condition [2]. Here we shall refer to it as the diagonal sequence condition. Thus a topological space X is said to satisfy the diagonal sequence condition if $x \in X$ and $(x_n^m)_{n=1}^{\infty}$ for each positive integer m is a sequence converging to x, it is possible to choose a sequence $(m(k))_{k=1}^{\infty}$ of distinct positive integers and a sequence $(j(k))_{k=1}^{\infty}$ of positive integers in such a way that the diagonal sequence $(x_{j(k)}^{m(k)})_{k=1}^{\infty}$ converges to x.

One can check easily that every metric space and more generally every first-countable space satisfies the diagonal sequence condition. Also note that sequential fan does not satisfy the diagonal sequence condition.

Almost disjoint sets:
Two sets A and B are said to be almost disjoint if their intersection is finite.

Pairwise almost disjoint family:
A pairwise almost disjoint family (abbr. as p. a. d. family) on a set X is a collection $\mathcal{F}$ of infinite subsets of X such that $A \cap B$ is finite for any two distinct members A, B in $\mathcal{F}$. For example, partition of N is a p. a. d. family on N.

A maximal p. a. d. family (abbr. as MAD family) on a set X is a p. a. d. family on X properly contained in no p. a. d. family on X. For example, $\mathcal{F} = \{O, E\}$, where O is the set of odd numbers and E is the set of even numbers, is a maximal p. a. d. family on N.

Ψ-space:
Let $\mathcal{F}$ be a p. a. d. family of infinite subsets of N. Let $\{\omega_F / F \in \mathcal{F}\}$ be a new set of distinct points and define $\Psi = N \cup \{\omega_F / F \in \mathcal{F}\}$ with the following topology: each subset of N is open; while $U \subset \Psi$ containing ω_F is open if and only if U contains all but finitely many points of F.

Easy application of Zorn's lemma shows that every such $\mathcal{F}$ is contained in a maximal p. a. d. family of infinite subsets of N. The spaces Ψ for such maximal p. a. d. families were first introduced by J. R. Isbell and considered in [3]. They appeared in [4] to provide an example of a compact, Hausdorff, sequential, non Fréchet-Urysohn space.

Frank Siwiec gave six examples which are countable spaces with exactly one nonisolated point [1]. Three of them are metrizable. Fourth one is the sequential fan and the other two contains a copy of sequential fan. These are nonhomeomorphic. Here we construct an example using Ψ-space having exactly one nonisolated point which contains no copy of sequential fan and hence it cannot be homeomorphic to sequential fan or two spaces like sequential fan.

For our convenience, we shall take Q the set of rational numbers and p. a. d. family on Q.

Example:

Ψ

Let $\Psi = Q \cup \{\omega_F / F \in \mathcal{F}\}$, and let $\tilde{\Psi}$ be the quotient space of Ψ by identifying all ω_F to one point, say ω. Our aim is to find a particular p. a. d. family $\mathcal{F}$ on Q such that $\tilde{\Psi}$ and sequential fan are not homeomorphic. We construct $\mathcal{F}$ in the following way.

Consider the unit interval $I = [0,1]$ in R with usual topology. For every $r \in [0,1]$, we can choose rational sequences (x_n^r) and (y_n^r) such that $A_r = \{x_n^r / n = 1,2,3,...\}$ and $B_r = \{y_n^r / n = 1,2,3,...\}$ are disjoint and both sequences converge to r. Set $\mathcal{G} = \{A_r / r \in [0,1]\} \cup \{B_r / r \in [0,1]\}$ which is uncountable p. a. d. family on Q. Then there exists a maximal p. a. d. family $\mathcal{M}$ on Q such that $\mathcal{G} \subset \mathcal{M}$. Let $\mathcal{F} = \mathcal{M} \setminus \{B_r / r \in [0,1]\}$. Now we show that under this family $\mathcal{F}$ the quotient space $\tilde{\Psi}$ is not homeomorphic to sequential fan. Suppose $\tilde{\Psi}$ contains a copy of sequential fan. Then there exists F_1 (say) in $\mathcal{F}$ such that we have a sequence (x_n^1) of points of F_1 which converges to some r_1 in $[0,1]$. Similarly, there exists F_2 (say) in $\mathcal{F}$ such that we have a sequence (x_n^2) of points of F_2 which converges to some r_2 in $[0,1]$. Continuing in this way, we get, because of compactness of $[0,1]$, a sequence (r_n) in $[0,1]$ which will converge to some r_0 in $[0,1]$. Now we can choose a point z_k in (x_n^k) for each k such that we have a diagonal sequence (z_k) in $\tilde{\Psi}$ with $G_{r_0} \cap A_{r_0} = \phi$, $G_{r_0} \cap B_{r_0} = \phi$, and $G_{r_0} \cap A_r$, $G_{r_0} \cap B_r$ are finite for all $r \in [0,1]$, where $G_{r_0} = \{z_k / k = 1,2,3,...\}$. Then $G_{r_0} \in \mathcal{F}$ and clearly (z_k) converges to r_0. Thus we get a diagonal sequence converging to ω, a contradiction to the fact that no diagonal sequence converges in a sequential fan. Thus $\tilde{\Psi}$ does not contain a copy of sequential fan, hence $\tilde{\Psi}$ and sequential fan are not homeomorphic.

References:

[1] Frank Siwiec, “Countable spaces having exactly one nonisolated point. I”, Proceedings of the AMS, Vol. 57, No. 2 (1976), 345-350.

[2] Peter J Nyikos, “Metrizability and the Fréchet-Urysohn property in topological groups”, Procedings of the AMS, Vol. 83 No.4 (1981), 793-801.

[3] L.Gillman and M. Jerison, “Rings of continuous functions”, Springer-Verlag, New York, (1976), 79

[4] S. P. Franklin, “Spaces in which sequences suffice-II”, Fund. Math. 61 (1967), 51-56.

[5] V. Kannan and M. Rajagopalan, “Hereditarily locally compact separable spaces”, Lecture Notes in Mathematics, Vol.719, (1979) Categorical Topology, 185-195.