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Cordial and 3-Equitable Graphs Induced by Duplication of Edge

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Abstract

We present here cordial and 3-equitable labeling for the graphs induced by duplicating an arbitrary edge of cycle and wheel.

Key words : Cordial labeling, 3-equitable labeling, Duplication of edge.
AMS **Subject classification number(2000):** 05C78.

1. INTRODUCTION

We begin with simple, finite, connected, undirected graph $G = (V(G), E(G))$. In the present work C_n denotes the cycle and $W_n = C_n + K_1$ denotes the wheel where $n \geq 3$. In W_n vertices correspond to C_n are called rim vertices and vertex corresponds to K_1 is called the apex vertex while edges correspond to C_n are called rim edges and edges joining K_1 and rim vertices are called spoke. Here $N(v)$ denotes the set of all neighboring vertices of v. For all other terminology and notations we follow Harary[6]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 *Duplication* of an edge $e = uv$ of graph G produces a new graph G_1 by adding an edge $e' = u'v'$ such that $N(u) = N(u')$ and $N(v) = N(v')$.

In other words an edge e' is said to be duplication of edge e if all the edges which are incident to e are now incident to e' also.

Definition 1.2 If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

According to Hegde[7] most interesting graph labeling problems have following three important characteristics.

1. a set of numbers from which the vertex labels are chosen;
2. a rule that assigns a value to each edge;
3. a condition that these values must satisfy.

The recent survey on graph labeling can be found in Gallian[5]. Vast amount of literature is available on different types of graph labeling. According to Beineke and Hegde[2] graph labeling serves as a frontier between number theory and structure of graphs.

Definition 1.3 Let $G = (V(G), E(G))$ be a graph. A mapping $f : V(G) \rightarrow \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the $label$ of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e)$=$|f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$,$e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.4 A binary vertex labeling of a graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *cordial* if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[3].

Many researchers have studied cordiality of graphs. e.g.Cahit [3] proved that tree is cordial. In the same paper he proved that K_n is cordial if and only if $n \leq 3$. Ho et al.[8] proved that unicyclic graph is cordial unless it is C_{4k+2}. Andar et al.[1] have discussed cordiality of multiple shells. Vaidya et al.[9, 10] have also discussed the cordiality of various graphs.

Definition 1.5 Let $G = (V(G), E(G))$ be a graph. A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called *ternary vertex labeling* of G and $f(v)$ is called *label of the vertex v* of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1, 2\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$, $v_f(2)$ be the number of vertices of G having labels 0, 1, 2 respectively under f and $e_f(0)$, $e_f(1)$, $e_f(2)$ be the number of edges having labels 0, 1, 2 respectively under f^*.

Definition 1.6 A ternary vertex labeling of a graph G is called a *3-equitable labeling* if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph G is *3-equitable* if it admits 3-equitable labeling.

The concept of 3-equitable labeling was introduced by Cahit[4]. Many researchers have studied 3-equatability of graphs. e.g.Cahit [4] proved that C_n is 3-equitable except $n \equiv 3(mod 6)$. In the same paper he proved that an Eulerian graph with number of edges congruent to $3(mod 6)$ is not 3-equitable. Youssef[12] proved that W_n is 3-equitable for all $n \geq 4$. Several results on 3-equitable labeling for some wheel related graphs in the context of vertex duplication are reported in Vaidya et al.[11]

2. Main Results

Theorem 2.1 Duplication of an arbitrary edge e_k of cycle C_n produces a cordial graph.

Proof Let C_n be the cycle with n vertices. Let $e_k = v_k v_{k+1}$ be the vertex of C_n. Let $e_k' = v_k' v_{k+1}'$ be the duplicated edge of e_k and G be the graph resulted due to duplication. To define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

Case 1: If $n \equiv 0(mod4)$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 0,3(mod4) \\ & = & 1;\ \text{if } i \equiv 1,2(mod4) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 0,3(mod4) \\ & = & 1;\ \text{if } i \equiv 1,2(mod4) \end{array}\right\} \text{for } n-k+2 \leq i \leq n$$

$$\begin{array}{r} f(v_k^{'}) = 0; \\ f(v_{k+1}^{'}) = 1; \text{if } k \neq n \\ f(v_{k-n+1}^{'}) = 1; \text{if } k = n \end{array}$$

Case 2: If $n \equiv 1(mod4)$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 0,3(mod4) \\ & = & 1;\ \text{if } i \equiv 1,2(mod4) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 0,3(mod4) \\ & = & 1;\ \text{if } i \equiv 1,2(mod4) \end{array}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$\begin{array}{r} f(v_{k-1}) = 0; \text{if } k \neq 1 \\ f(v_{k+n-1}) = 0; \text{if } k = 1 \\ f(v_k^{'}) = 0; \\ f(v_{k+1}^{'}) = 1; \text{if } k \neq n \\ f(v_{k-n+1}^{'}) = 1; \text{if } k = n \end{array}$$

Case 3: If $n \equiv 2(mod4)$

$$\begin{array}{r} f(v_k) = 1; \\ f(v_{k+1}) = 0; \text{if } k \neq n \\ f(v_{k-n+1}) = 0; \text{if } k = n \end{array}$$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 1,2(mod4) \\ & = & 1;\ \text{if } i \equiv 0,3(mod4) \end{array}\right\} \text{for } 3 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 1,2(mod4) \\ & = & 1;\ \text{if } i \equiv 0,3(mod4) \end{array}\right\} \text{for } n-k+2 \leq i \leq n$$

$$\begin{array}{r} f(v_k^{'}) = 0; \\ f(v_{k+1}^{'}) = 1; \text{if } k \neq n \\ f(v_{k-n+1}^{'}) = 1; \text{if } k = n \end{array}$$

Case 4: If $n \equiv 3(mod4)$

$$f(v_k) = 1;$$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 2,3(mod4) \\ & = & 1;\ \text{if } i \equiv 0,1(mod4) \end{array}\right\} \text{for } 2 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 2,3(mod4) \\ & = & 1;\ \text{if } i \equiv 0,1(mod4) \end{array}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$\begin{array}{r} f(v_{k-1}) = 1; \text{if } k \neq 1 \\ f(v_{k+n-1}) = 1; \text{if } k = 1 \\ f(v_k^{'}) = 1; \\ f(v_{k+1}^{'}) = 0; \text{if } k \neq n \\ f(v_{k-n+1}^{'}) = 0; \text{if } k = n \end{array}$$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in *Table 1*(where $n = 4a + b$ and $a, b \in N \cup \{0\}$). That is, G admits cordial labeling.

b	**Vertex Condition**	**Edge Condition**
0,2	$v_f(0) = v_f(1) = \frac{n+2}{2}$	$e_f(0) + 1 = e_f(1) = \frac{n+4}{2}$
1	$v_f(0) = v_f(1) + 1 = \frac{n+3}{2}$	$e_f(0) = e_f(1) = \frac{n+3}{2}$
3	$v_f(0) + 1 = v_f(1) = \frac{n+3}{2}$	$e_f(0) = e_f(1) = \frac{n+3}{2}$

Table 1

Illustration 2.2 Consider C_{10} and duplicate e_2. The cordial labeling is as shown in *Figure 1*

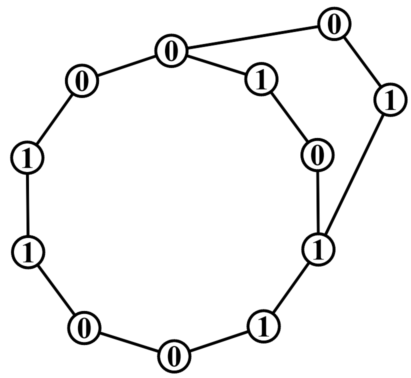

Figure 1

Theorem 2.3 Duplication of an arbitrary edge e_k of wheel W_n produces a cordial graph.

Proof Consider the wheel $W_n = C_n + K_1$. Let $v_1, v_2, \ldots, v_n$ be the rim vertices of W_n, c be the apex vertex of W_n and G be the graph obtained by duplicating either rim edge or spoke edge of W_n. Let $e_k^{'}$ be the duplicated edge of e_k. To define binary vertex labeling $f : V(G) \to \{0, 1\}$ we consider the following cases.

Case 1: Duplication of an arbitrary rim edge e_k, where $k \in N, 1 \leq k \leq n$
Subcase 1: If $n \equiv 0(mod4)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,1(mod4) \\ &= 1;\ \text{if } i \equiv 2,3(mod4) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$
$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,1(mod4) \\ &= 1;\ \text{if } i \equiv 2,3(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = f(v_k^{'}) = 0;$$
$$f(v_{k+1}^{'}) = 1; \text{if } k \neq n$$
$$f(v_{k-n+1}^{'}) = 1; \text{if } k = n$$

Subcase 2: If $n \equiv 1, 2(mod4)$
$$f(v_k) = 0;$$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,1(mod4) \\ &= 1;\ \text{if } i \equiv 2,3(mod4) \end{aligned}\right\} \text{for } 2 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 0,1(mod4) \\ & = & 1;\ \text{if } i \equiv 2,3(mod4) \end{array}\right\} \text{for } n-k+2 \leq i \leq n$$

$$\begin{array}{r} f(c) = 0; \\ f(v_k^{'}) = 1; \\ f(v_{k+1}^{'}) = 1; \text{if } k \neq n \\ f(v_{k-n+1}^{'}) = 1; \text{if } k = n \end{array}$$

Subcase 3: If $n \equiv 3(mod4)$

$$\begin{array}{rcll} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 1,2(mod4) & \multirow{2}{*}{$\Big\}$ for $1 \leq i \leq n-k+1$} \\ & = & 1;\ \text{if } i \equiv 0,3(mod4) & \\ f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 1,2(mod4) & \multirow{2}{*}{$\Big\}$ for $n-k+2 \leq i \leq n$} \\ & = & 1;\ \text{if } i \equiv 0,3(mod4) & \end{array}$$

$$\begin{array}{r} f(c) = 0; \\ f(v_k^{'}) = 1; \\ f(v_{k+1}^{'}) = 1; \text{if } k \neq n \\ f(v_{k-n+1}^{'}) = 1; \text{if } k = n \end{array}$$

Case 2: Duplication of an arbitrary spoke edge $e_k = cv_k$, where $k \in N, n+1 \leq k \leq 2n$
Subcase 1: If $n \equiv 0(mod4)$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 0,3(mod4) \\ & = & 1;\ \text{if } i \equiv 1,2(mod4) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$
$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 0,3(mod4) \\ & = & 1;\ \text{if } i \equiv 1,2(mod4) \end{array}\right\} \text{for } n-k+2 \leq i \leq n$$

$$\begin{array}{r} f(c) = f(c^{'}) = 0; \\ f(v_k^{'}) = 1; \end{array}$$

Subcase 2: If $n \equiv 1(mod4)$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 0,3(mod4) \\ & = & 1;\ \text{if } i \equiv 1,2(mod4) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$
$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 0,3(mod4) \\ & = & 1;\ \text{if } i \equiv 1,2(mod4) \end{array}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$\begin{array}{r} f(v_{k-1}) = 0; \text{if } k \neq 1 \\ f(v_{k+n-1}) = 0; \text{if } k = 1 \\ f(c) = 0; \\ f(c^{'}) = 1; \\ f(v_k^{'}) = 1; \end{array}$$

Subcase 3: If $n \equiv 2(mod4)$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 1,2(mod4) \\ & = & 1;\ \text{if } i \equiv 0,3(mod4) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$
$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 1,2(mod4) \\ & = & 1;\ \text{if } i \equiv 0,3(mod4) \end{array}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$\begin{array}{r} f(v_{k-1}) = 1; \text{if } k \neq 1 \\ f(v_{k+n-1}) = 1; \text{if } k = 1 \end{array}$$

$f(c) = 0;$
$f(c') = 1;$
$f(v'_k) = 1;$

Subcase 4: If $n \equiv 3(mod4)$

$f(v_k) = 0;$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 0,1(mod4) \\ & = & 1;\ \text{if } i \equiv 2,3(mod4) \end{array}\right\} \text{for } 2 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 0,1(mod4) \\ & = & 1;\ \text{if } i \equiv 2,3(mod4) \end{array}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$f(v_{k-1}) = 0;$ if $k \neq 1$
$f(v_{k+n-1}) = 0;$ if $k = 1$
$f(c) = 0;$
$f(c') = 1;$
$f(v'_k) = 1;$

The labeling pattern defined above covers all the possibilities. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in *Table 2*(where $n = 4a + b$ and $a \in N \cup \{0\}$). That is, G admits cordial labeling.

b	**Vertex Condition**	**Edge Condition**
Duplication of a rim edge		
0	$v_f(0) = v_f(1) + 1 = \frac{n+4}{2}$	$e_f(0) = e_f(1) + 1 = n + 3$
1,3	$v_f(0) = v_f(1) = \frac{n+3}{2}$	$e_f(0) = e_f(1) + 1 = n + 3$
2	$v_f(0) + 1 = v_f(1) = \frac{n+4}{2}$	$e_f(0) + 1 = e_f(1) = n + 3$
Duplication of a spoke edge		
0	$v_f(0) = v_f(1) + 1 = \frac{n+4}{2}$	$e_f(0) = e_f(1) = \frac{3n+2}{2}$
1,3	$v_f(0) = v_f(1) = \frac{n+3}{2}$	$e_f(0) = e_f(1) + 1 = \frac{3n+3}{2}$
2	$v_f(0) + 1 = v_f(1) = \frac{n+4}{2}$	$e_f(0) = e_f(1) = \frac{3n+2}{2}$

Table 2

Illustration 2.4 Consider W_4 and duplicate spoke edge e_6. The cordial labeling is as shown in *Figure 2*

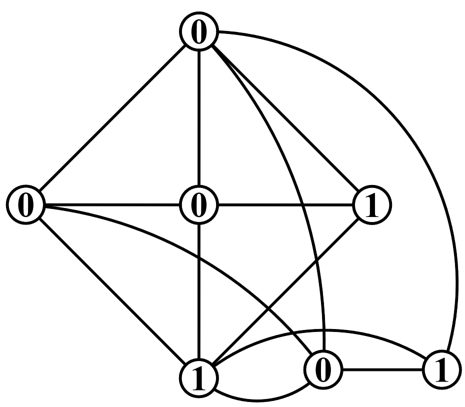

Figure 2

Theorem 2.5 Duplication of an arbitrary edge e_k of cycle C_n produces a 3-equitable graph.

Proof Let C_n be the cycle with n vertices. Let $e_k = v_k v_{k+1}$ be the vertex of C_n. Let $e_k^{'} = v_k^{'} v_{k+1}^{'}$ be the duplicated edge of e_k and G be the graph resulted due to duplication. To define ternary vertex labeling $f : V(G) \rightarrow \{0, 1, 2\}$ we consider following cases.

Case 1: If $n \equiv 1, 2, 3, 4(mod6)$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 0, 3(mod6) \\ & = & 1;\ \text{if } i \equiv 1, 2(mod6) \\ & = & 2;\ \text{if } i \equiv 4, 5(mod6) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 0, 3(mod6) \\ & = & 1;\ \text{if } i \equiv 1, 2(mod6) \\ & = & 2;\ \text{if } i \equiv 4, 5(mod6) \end{array}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$\begin{array}{r} f(v_{k-1}) = 0;\ \text{if } k \neq 1 \\ f(v_{k+n-1}) = 0;\ \text{if } k = 1 \\ f(v_k^{'}) = 2;\ \text{if } n \equiv 2, 3, 4(mod6) \\ f(v_k^{'}) = 1;\ \text{if } n \equiv 1(mod6) \\ f(v_{k+1}^{'}) = 2;\ \text{if } k \neq n \\ f(v_{k-n+1}^{'}) = 2;\ \text{if } k = n \end{array}$$

Case 2: If $n \equiv 0(mod6)$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 1, 4(mod6) \\ & = & 1;\ \text{if } i \equiv 2, 3(mod6) \\ & = & 2;\ \text{if } i \equiv 0, 5(mod6) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 1, 4(mod6) \\ & = & 1;\ \text{if } i \equiv 2, 3(mod6) \\ & = & 2;\ \text{if } i \equiv 0, 5(mod6) \end{array}\right\} \text{for } n-k+2 \leq i \leq n$$

$$\begin{array}{r} f(v_k^{'}) = 0; \\ f(v_{k+1}^{'}) = 1;\ \text{if } k \neq n \\ f(v_{k-n+1}^{'}) = 1;\ \text{if } k = n \end{array}$$

Case 3: If $n \equiv 5(mod6)$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 2, 5(mod6) \\ & = & 1;\ \text{if } i \equiv 0, 1(mod6) \\ & = & 2;\ \text{if } i \equiv 3, 4(mod6) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 2, 5(mod6) \\ & = & 1;\ \text{if } i \equiv 0, 1(mod6) \\ & = & 2;\ \text{if } i \equiv 3, 4(mod6) \end{array}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$\begin{array}{r} f(v_{k-1}) = 1;\ \text{if } k \neq 1 \\ f(v_{k+n-1}) = 1;\ \text{if } k = 1 \end{array}$$

$$f(v_k') = 0;$$
$$f(v_{k+1}') = 0; \text{ if } k \neq n$$
$$f(v_{k-n+1}') = 0; \text{ if } k = n$$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in *Table 3*(where $n = 6a + b$ and $a \in N \cup \{0\}$). That is, G admits 3-equitable labeling.

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0) = v_f(1) = v_f(2) + 1 = \frac{n+3}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{n+3}{3}$
1	$v_f(0) = v_f(1) = v_f(2) = \frac{n+2}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = \frac{n+5}{3}$
2	$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \frac{n+4}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) = \frac{n+4}{3}$
3	$v_f(0) + 1 = v_f(1) = v_f(2) = \frac{n+3}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{n+3}{3}$
4	$v_f(0) = v_f(1) = v_f(2) = \frac{n+2}{3}$	$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = \frac{n+5}{3}$
5	$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = \frac{n+4}{3}$	$e_f(0) = e_f(1) = e_f(2) + 1 = \frac{n+4}{3}$

Table 3

Illustration 2.6 Consider C_9 and duplicate edge e_3. The corresponding 3-equitable labeling is shown in *Figure 3*

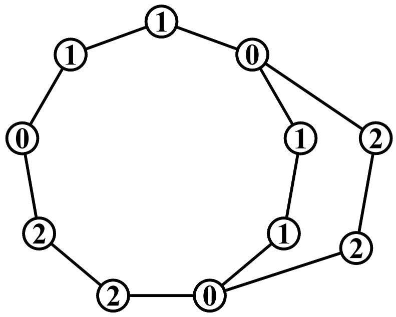

Figure 3

Theorem 2.7 Duplication of an arbitrary edge e_k of wheel W_n produces a 3-equitable graph.

proof Consider the wheel $W_n = C_n + K_1$. Let $v_1, v_2, \ldots, v_n$ be the rim vertices of W_n, c be the apex vertex of W_n and G be the graph obtained by duplicating either rim edge or spoke edge of W_n. Let e_k' be the duplicated edge of e_k. To define ternary vertex labeling $f : V(G) \rightarrow \{0, 1, 2\}$ we consider following cases.

Case 1: Duplication of arbitrary rim edge e_k, where $k \in N, 1 \leq k \leq n$
Subcase 1: If $n \equiv 0, 5(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1, 4(mod6) \\ &= 1;\ \text{if } i \equiv 2, 3(mod6) \\ &= 2;\ \text{if } i \equiv 0, 5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$
$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1, 4(mod6) \\ &= 1;\ \text{if } i \equiv 2, 3(mod6) \\ &= 2;\ \text{if } i \equiv 0, 5(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$f(c) = 0;$
$f(v_k^{'}) = 2;$

$$\left.\begin{aligned} f(v_{k+1}) &= 1;\ \text{if } n \equiv 0(mod6) \\ &= 2;\ \text{if } n \equiv 5(mod6) \end{aligned}\right\} \text{for } k \neq n$$

$$\left.\begin{aligned} f(v_{k-n+1}) &= 1;\ \text{if } n \equiv 0(mod6) \\ &= 2;\ \text{if } n \equiv 5(mod6) \end{aligned}\right\} \text{for } k = n$$

Subcase 2: If $n \equiv 1, 2, 3, 4(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0, 3(mod6) \\ &= 1;\ \text{if } i \equiv 1, 2(mod6) \\ &= 2;\ \text{if } i \equiv 4, 5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0, 3(mod6) \\ &= 1;\ \text{if } i \equiv 1, 2(mod6) \\ &= 2;\ \text{if } i \equiv 4, 5(mod6) \end{aligned}\right\} \begin{array}{l} \text{for } n-k+2 \leq i \leq n-1 \text{ and } n \equiv 1(mod6) \\ \text{for } n-k+2 \leq i \leq n \text{ and } n \equiv 2, 3, 4(mod6) \end{array}$$

$f(v_{k-1}) = 0$; if $n \equiv 1(mod6)$ and $k \neq 1$
$f(v_{k+n-1}) = 0$; if $n \equiv 1(mod6)$ and $k = 1$
$f(c) = 0$; if $n \equiv 2, 4(mod6)$
$f(c) = 2$; if $n \equiv 1, 3(mod6)$
$f(v_k^{'}) = 0$; if $n \equiv 2(mod6)$
$f(v_k^{'}) = 1$; if $n \equiv 1(mod6)$
$f(v_k^{'}) = 2$; if $n \equiv 3, 4(mod6)$

$$\left.\begin{aligned} f(v_{k+1}) &= 0;\ \text{if } n \equiv 3, 4(mod6) \\ &= 2;\ \text{if } n \equiv 1, 2(mod6) \end{aligned}\right\} \text{for } k \neq n$$

$$\left.\begin{aligned} f(v_{k-n+1}) &= 0;\ \text{if } n \equiv 3, 4(mod6) \\ &= 2;\ \text{if } n \equiv 1, 2(mod6) \end{aligned}\right\} \text{for } k = n$$

Case 2: Duplication of arbitrary spoke edge $e_{n+k} = cv_k$, where $k \in N, 1 \leq k \leq n$
Subcase 1: If $n \equiv 0, 5(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0, 3(mod6) \\ &= 1;\ \text{if } i \equiv 1, 2(mod6) \\ &= 2;\ \text{if } i \equiv 4, 5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0, 3(mod6) \\ &= 1;\ \text{if } i \equiv 1, 2(mod6) \\ &= 2;\ \text{if } i \equiv 4, 5(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$f(c) = 0;$
$f(c^{'}) = 2;$
$f(v_k^{'}) = 1$; if $n \equiv 0(mod6)$
$f(v_k^{'}) = 0$; if $n \equiv 5(mod6)$

Subcase 2: If $n \equiv 1(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1, 4(mod6) \\ &= 1;\ \text{if } i \equiv 2, 3(mod6) \\ &= 2;\ \text{if } i \equiv 0, 5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 2;$$
$$f(c^{'}) = 2;$$
$$f(v_k^{'}) = 1;$$

Subcase 3: If $n \equiv 2(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 0;$$
$$f(c^{'}) = 2;$$
$$f(v_k^{'}) = 1;$$

Subcase 4: If $n \equiv 3(mod6), n \neq 3$

$$f(v_k) = 1;$$
$$f(v_{k+1}) = 0; \text{if } k+1 \leq n$$
$$f(v_{k-n+1}) = 0; \text{if } k+1 > n$$
$$f(v_{k+2}) = 2; \text{if } k+2 \leq n$$
$$f(v_{k-n+2}) = 2; \text{if } k+2 > n$$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } 4 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 0;$$
$$f(c^{'}) = 2;$$
$$f(v_k^{'}) = 1;$$

If $n = 3$ the labeling starting from v_k is $0, 2, 2$ for rim vertices, labeling of apex vertex 0 and labeling of vertices $v_k^{'}$ and $c^{'}$ is 1.

Subcase 5: If $n \equiv 4(mod6), n \neq 4$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 1,2(mod6) \\ &= 2;\ \text{if } i \equiv 4,5(mod6) \end{aligned}\right\} \text{for } 4 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 1,2(mod6) \\ &= 2;\ \text{if } i \equiv 4,5(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n-4$$

$f(v_{k+i-n-1}) = 2$; if $n-3, n-2$
$f(v_{k+i-n-1}) = 1$; if $n-1, n$
$f(c) = 0$;
$f(c^{'}) = 0$;
$f(v_k^{'}) = 1$;

If $n = 4$ the labeling starting from v_k is $0, 2, 2, 0$ for rim vertices, labeling of apex vertex 2 and labeling of vertices $v_k^{'}$ and $c^{'}$ is 1.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in *Table 4*(where $n = 6a + b$ and $a \in N \cup \{0\}$). That is, G admits 3-equitable labeling.

b	**Vertex Condition**	**Edge Condition**
	Duplication of a rim edge	
0	$v_f(0) = v_f(1) = v_f(2) = \frac{n+3}{3}$	$e_f(0) = e_f(1) = e_f(2) + 1 = \frac{2n+6}{3}$
1	$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \frac{n+5}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = \frac{2n+7}{3}$
2	$v_f(0) = v_f(1) = v_f(2) + 1 = \frac{n+4}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{2n+5}{3}$
3	$v_f(0) = v_f(1) = v_f(2) = \frac{n+3}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) = \frac{2n+6}{3}$
4	$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = \frac{n+5}{3}$	$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = \frac{2n+7}{3}$
5	$v_f(0) = v_f(1) + 1 = v_f(2) = \frac{n+4}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{2n+5}{3}$
	Duplication of a spoke edge	
0	$v_f(0) = v_f(1) = v_f(2) = \frac{n+3}{3}$	$e_f(0) = e_f(1) = e_f(2) + 1 = n + 1$
1	$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \frac{n+5}{3}$	$e_f(0) = e_f(1) = e_f(2) + 1 = n + 1$
2	$v_f(0) = v_f(1) + 1 = v_f(2) = \frac{n+4}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) = n + 1$
3	$v_f(0) = v_f(1) = v_f(2) = \frac{n+3}{3}$	$e_f(0) = e_f(1) + 1 = e_f(2) = n + 1$
4	$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = \frac{n+5}{3}$	$e_f(0) = e_f(1) + 1 = e_f(2) = n + 1$
5	$v_f(0) = v_f(1) + 1 = v_f(2) = \frac{n+4}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) = n + 1$

Table 4

Illustration 2.8 Consider W_5 and duplicate edge e_1. The corresponding 3-equitable labeling is shown in *Figure 4*

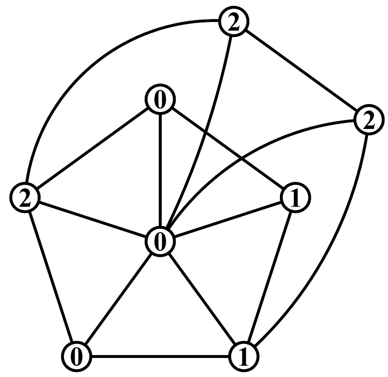

Figure 4

3. Concluding Remarks

The present work corresponds to cordial and 3-equitable labeling of some cycle related graphs. This approach is novel as it provides cordial and 3-equitable labeling in the context of edge operation.

Further Scope of Research

Similar investigations can be carried out in the context of different graph labeling techniques and for various standard graphs.

References

[1] Andar, M., Boxwala, S. and Limaye, N. B. (2002): A Note on cordial labeling of multiple shell, Trends Math., 77-80.

[2] Beineke,L.W. and Hegde,S.M.(2001): Strongly Multiplicative graphs, Discuss. Math. Graph Theory, **21**, 63-75. (http://www.discuss.wmie.uz.zgora.pl/gt/index.php)

[3] Cahit, I. (1987): Cordial Graphs: A weaker version of graceful and harmonious Graphs, Ars Combinatoria, **23**, 201-207.

[4] Cahit, I. (1990): On cordial and 3-equitable labelings of graphs, Util. Math., **37**, 189-198.

[5] Gallian, J. A. (2010): A dynamic survey of graph labeling, The Electronics Journal of Combinatorics, **17** $\sharp DS6$. (http://www.combinatorics.org/Surveys/index.html)

[6] Harary, F. (1972): Graph theory, Addison Wesley, Reading, Massachusetts.

[7] Hegde, S. M. (2008): On Multiplicative Labelings of a Graph, Labeling of Discrete Structures and applications, Narosa Publishing House, New Delhi, 83-96.

[8] Ho, Y. S., Lee, S. M. and Shee, S. C. (1989): Cordial labeling of unicyclic graphs and generalized Petersen graphs, Congress. Numer., **68**, 109-122.

[9] Vaidya, S. K., Ghodasara, G. V., Srivastav, S. and Kaneria, V. J. (2008): Some new cordial graphs, Int. J. of scientific comp. **2(1)**, 81-92.

[10] Vaidya, S. K., Ghodasara, G. V., Srivastav, S. and Kaneria, V. J. (2008): Cordial labeling for cycle with one chord and its related graphs, Indian J. of Math. and Math.Sci **4(2)**, 145-156.

[11] Vaidya, S. K., Dani, N. A., Kanani, K. K. and Vihol, P. L. (2009): Some wheel related 3-Equitable Graphs in the context of vertex duplication, Advance Appl. in Discrete Math., **4(1)**, 71-85. (www.pphmj.com/journals/aadm.html)

[12] Youssef, M. Z. (2003): A necessary condition on k-equitable labelings, Util. Math., **64**, 193-195.