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HAMILTONIAN NUMBER OF (a,b) DEGREE GRAPHS AND TOTAL GRAPH OF A TREE

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Abstract

In this paper, the authors develop a formula for counting of total number of paths of length m ($m \leq q$) of a tree T on q edges, the longest cycle C in the (a,b) degree graph. Also as the longest cycle C in the (a,b) degree graph is always a dominating cycle, we derive a formula for the Hamiltonian number $h(G)$ of (a,b) degree graph and total graph of a tree T.

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Goodman and Hedetniemi [6, 7] first begin the journey on Hamiltonian walk in a connected graph $G(V, E)$. Hamiltonian number of a graph G is defined as a length of the minimal closed spanning walk in G and it is denoted by $h(G)$. For any graph G of order $n \geq 3$, $h(G) = n$ if and only if G is Hamiltonian. Clearly 0-Hamiltonian graph is Hamiltonian and Peterson graph is 1-Hamiltonian graph. Hamiltonian number is used as a tool to measure the graph, how much it is lacking to be Hamiltonian. In a connected graph G of order n, adding k-edges ($k \leq n-2$) suitably changes the graph to be Hamiltonian. Then the graph G is k-Hamiltonian,($k \geq 0$) and $h(G) = n + k$. Further studies in Hamiltonian walks were carried out by J.C.Bermond [1],T. Asano,T. Nishizeki,T. Watanabe [10, 11], and P. Vacek [9] respectively. A bipartite graph $G[X, Y]$ is non-Hamiltonian iff $|X| \geq |Y|$. If K and L are subsets of $V(G)$ or subgraphs of G then $N_K(L)$ is the set of vertices in K adjacent to some vertex in L. Moreover $|N_K(v)| = d_K(v)$ [4]. For the component H of $G-C$ we take $N_C(H) = \{x_1, x_2, ..., x_t\}$ according to the traversing way of C (subscripts are consider to modulo t). For $v \in V(C)$, where C is a cycle in a connected graph G, we denote by v^+ its successor and by v^- its predecessor on C.

With this background, in this paper we establish the behavior of the longest cycle in (a,b) degree graph defined below. It is observed that the minimum vertex deletion in a graph G leads Hamiltonian graph and hence we apply the same to obtain the Hamiltonian number of (a,b) degree graphs which can be utilized to find the closed spanning walk of minimum length in the network analysis. Also we find the Hamiltonian number of total graph of a tree and formula for counting of total number of paths of length m for a given tree T on q edges($m \leq q$).Throughout this paper graph means simple connected graph.

1 Basic Definitions

In this section, we present basic definitions and priliminary results which will be useful to derive the main results.

Definition 1.1 *[2] A cycle C of a graph G is a dominating cycle if every edge of G has at least one end in C.*

Definition 1.2 *[5] The Total graph $\Im(G)$ of a graph G has $V(G)$ and $E(G)$ as its vertex set and two vertices in $\Im(G)$ are adjacent if and only if they are adjacent or incident in G.*

Definition 1.3 *Let $a, b \geq 2$ be two integers. A simple connected bipartite graph $G[X, Y]$ is said to be (a, b) degree graph if $deg(x) = a$ and $deg(y) = b$ $\forall x \in X$ and $\forall y \in Y$. Note that in (a, b) degree graph $|V(G)| = (a+b)m$ and $|E(G)| = abm$, where $m = \frac{|Y|}{a} = \frac{|X|}{b} = \frac{|Y|}{d(x)} = \frac{|Y|}{d(y)}$.*

Proposition 1.4 *[2] Let G[X, Y] be a bipartite graph without isolated vertices, such that $d(x) \geqslant d(y)$, $\forall xy \in E(G)$,where $x \in X$ and $y \in Y$. Then $|X| \leqslant |Y|$, with equality if and only if $d(x) = d(y)$ $\forall xy \in E(G)$.*

Lemma 1.5 *[10] If a connected graph G of order n has a cycle C of length $l(C)$, then $h(G) \leqslant 2n - l(C)$.*

Theorem 1.6 *[3] For every connected graph G of order $n \geqslant 2$,$n \leqslant h(G) \leqslant 2n-2$. Moreover, (i) $h(G) = 2n - 2$ iff G is a tree*
(ii) For every pair n, k of integers with $3 \leqslant n \leqslant k \leqslant 2n-2$,there exists a connected graph G of order n having $h(G) = k$.

Theorem 1.7 *[9], [3]Let G be a connected graph and $B_1, B_2, \ldots, B_k$ be the blocks of G.Then $h(G) = \sum_{i=1}^{k} h(B_i)$.*

Remark 1.8 *[12] If a connected graph G contains an edge e such that $(G - e)$ is connected, then $h(G) \leq h(G - e)$.*

2 Main Results

In this section we derive some significant results and formula for counting of total number of paths of length m of a tree T on q edges and on Hamiltonian number of (a, b) degree graphs.

Theorem 2.1 *Let T be a tree on q edges with $(2 < m \leqslant q)$ and $N_T(P_m)$: = Total number of paths of length m in the tree T. Then*

$$N_T(P_m) = \sum_{d(u,v)=m-2} [d_T(u) - 1]\,[d_T(v) - 1]. \quad (1)$$

Proof 2.2 *We shall prove the identity (1) by induction. When $|V(T)| = 4, m = 3, q = 3$, the tree T must be a path of length three or a star $K_{1,3}$ and hence T satisfies identity (1).*
Assume the identity (1) holds for tree T_1 with $|V(T_1)| = q,\ 2 < m \leqslant q-1$. Therefore by hypothesis, for a subtree $T_1 = T - v$, where v is a pendent vertex of T,

$$N_{T_1}(P_m) = \sum_{d(x,y)=m-2} [d_{T_1}(x) - 1]\,[d_{T_1}(y) - 1]. \tag{2}$$

Since $uv \notin E(T_1)$ and $uv \in E(T)$,
$N_{T_1}(P_m)$ can be expressed as

$$N_{T_1}(P_m) = [d_{T_1}(u) - 1]\,[d_{T_1}(u_1) - 1] + [d_{T_1}(u) - 1]\,[d_{T_1}(u_2) - 1]$$

$$+ \cdots + [d_{T_1}(u) - 1]\,[d_{T_1}(u_k) - 1] + \textit{other terms in } T_1$$

, where $d(u, u_i) = m - 2, (i = 1, 2, 3, \cdots, k)$.

$$\textit{Since } \mathrm{d}_T(x) = \mathrm{d}_{T_1}(x) \quad \forall \mathrm{x} \in \mathrm{V}(T_1)\backslash u \text{ and } \quad \mathrm{d}_T(u) = \mathrm{d}_{T_1}(u) + 1,$$

$$N_{T_1}(P_m) = [(d_T(u) - 1) - 1]\,[d_T(u_1) - 1]$$
$$+ [(d_T(u) - 1) - 1]\,[d_T(u_2) - 1] + \cdots$$
$$+ [(d_T(u) - 1) - 1]\,[d_T(u_k) - 1] + \textit{other terms in } T$$

$$i.e., N_{T_1}(P_m) = N_T(P_m) - \sum_{i=1}^{k} [d_T(u_i) - 1]$$

since,

$$\sum_{i=1}^{k} [d_T(u_i) - 1] = \sum_{\substack{i=1 \\ d(u,u_i)=m-2}} [(d_{T_1}(u) + 1) - 1]\,[d_{T_1}(u_i) - 1]$$

$$- \sum_{\substack{i=1 \\ d(u,u_i)=m-2}} [d_{T_1}(u) - 1]\,[d_{T_1}(u_i) - 1],$$

we find

$$N_T(P_m) = N_{T_1}(P_m) + \sum_{\substack{i=1 \\ d(u,u_i)=m-2}}^{k} [(d_{T_1}(u) + 1) - 1]\,[d_{T_1}(u_i) - 1]$$

$$- \sum_{\substack{i=1 \\ d(u,u_i)=m-2}}^{k} [d_{T_1}(u) - 1]\,[d_{T_1}(u_i) - 1] \tag{3}$$

Now the proof of identity (1) follows by (2) and (3) .

Theorem 2.3 *If T is a tree on q edges ($1 < m \leqslant q$), then*

$$N_T(P_m) = \frac{1}{2} \sum_{d(u,v)=m-1} [(d_T(u) - 1) + (d_T(v) - 1)].$$

Proof 2.4 *The proof follows from theorem 2.1.*

Remark 2.5 *For* $m = 3$, $N_T(P_3) = \sum\limits_{d(u,v)=1} [d_T(u) - 1]\,[d_T(v) - 1]$
i.e., Counting of total number of paths of length three in tree T. *The above formula is true for any simple triangle free graphs.*
In General formula for $N_G(P_m)$ *given in (1) is true for any simple* C_i *free graphs (*$i = 3, 4, 5, ..., m$*) where* C_i *is cycle of length* i.

Lemma 2.6 *Let* $G[X, Y]$ *be a* $(2, b)$ *degree graph (*$b > 2$*) and* $u, v \in V(C)$ *such that* $d_G(u), d_G(v) = b$ *and* $d_C(u, v) = 2$. *If* $N_{G-C}(u^-), N_{G-C}(v^+)$ *are* C*- saturated, then* C *is the longest cycle in* G.

Proof 2.7 *Suppose* $N_{G-C}(u^-), N_{G-C}(v^+)$ *are* C*-unsaturated. Since* G *is 2-connected, then one can construct longer cycle* C_1 *than* C *through a traversal from* u *to* v *in* $G - C$. *Suppose if* $u, v \in V(C_1)$ *and* $N_{G-C_1}(u^-), N_{G-C_1}(v^+)$ *are* C_1 *- unsaturated, then we can continue the process up to finding a cycle* C_k $(k < (2m + bm), m > 1)$, *such that no paths exist between* u *and* v *in* $G - C_k$. *Then* C_k *is the required longest cycle* C *in* G. *Therefore any arbitrary component* H *of* $G - C$ *has isolated vertices and hence* C *is the dominating cycle in* G.

Lemma 2.8 *Let* $G[X, Y]$ *be a* (a, b) *degree graph (*$b > a, a > 2$*)and* $e = \{u, v\} \in E(C)$. *Then* C *is a longest cycle in* G *if and only if* $N_{G+e-C}(P_3) = 0$.

Proof 2.9 *Suppose that there exists an edge* $e = \{u, v\} \in E(C)$ *such that* $N_{G+e-C}(P_3) \neq 0$. *Since* G *is 2-connected, there exists a path between* u *and* v *in* G. *So we can construct a longer cycle* C_1 *than* C *in* G *which is a contradiction to our assumption that* C *is the longest cycle in* G. *Similarly we can prove the other part(*$G - C$ *has isolated vertices).*

Theorem 2.10 *The longest cycle in the (a,b) degree graph is always a dominating cycle.*

Proof 2.11 *Using Lemmas 2.6 and 2.8,* $G - C$ *has isolated vertices and hence we can conclude that the longest cycle in the (a,b) degree graph is always a dominating cycle.*

Remark 2.12 *If* C *is a dominating cycle in a* (a, b) *degree graph* $G[X, Y]$, *then the Hamiltonian number* $h(G) = |V(G)| + |V(G - C)| = |V(C)| + 2\,|V(G - C)| = 2bm$

Remark 2.13 *Let* $G[X, Y]$ *be a* (a, b) *degree bipartite graph with* n *vertices and* $a, b > 2$. *If* $G - \min\{v_1, v_2, ..., v_k\}$ *is Hamiltonian(*$k < n - 2$*) ,then the Hamiltonian number* $h(G) = n + k$.

Similarly the following propositions give Hamiltonian number for a total graph $\Im(T)$ of a tree T for the star tree $K_{1,n-1}$ and Path of length $n - 1$ of order n.

Proposition 2.14 *Let* $K_{1,n-1}$ *be a star of order* n. $|V\,[\Im(K_{1,n-1})]| = 2n - 1$ *The Hamiltonian Number of* $\Im(K_{1,n-1})$ *is* $h\,[\Im\,(K_{1,n-1})] = 3n - 4$

Proof 2.15 *Since every edge is adjacent to all other edges in* $K_{1,n-1}$, *the total graph* $\Im(K_{1,n-1})$ *has a cycle of length* $n-1$ *and hence the proof follows* $h\,[\Im(K_{1,n-1})] = (2n-1)+[(n-1)-2] = 3n - 4$

Remark 2.16 *The longest cycle in* $\Im(K_{1,n-1})$ *is a dominating cycle.*

Proposition 2.17 *Let* P *be a Path of length* $n - 1$ *and* $|V\,([\Im(P)])| = 2n - 1$. *Then* $\Im(P)$ *is Hamiltonian.*

Remark 2.18 *If* T *is a tree of order* n *(neither a star nor a path), then the Hamiltonian number* $h\,[\Im\,(T)]$ *satisfies the following inequality.*

$$h\,[\Im(T)] \leq h\,[\Im(K_{1,n-1})] \Rightarrow h\,[\Im(T)] \leq 3n - 4.$$

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