Mathematics Today Vol.27(June-2011)(Proc. of Maths Meet – 2011) 6-17
ISSN 0976-3228

Using a Systematic Group Theoretic Method to Solve Flow of an Incompressible Second-Order Fluid past a Stretching Sheet

Adnan K. Al-Salihi, A. H. Hasmani# and M. G. Timol*
Department of Mathematics, Sardar Patel University,
Vallabh Vidyanagar 388120, Gujarat, India.
adnank2010@yahoo.com
ahhasmani@gmail.com

*Department of Mathematics, V.N. South Gujarat University
Magdalla Road, Surat-395007, Gujarat, India.
mgtimol@gmail.com

Abstract
A solution methodology for a system of partial differential equations based on extension systematic group theoretic method was presented by Al-Salihi et al [3]. The method is an application of two-parameter group transformation which reduces the number of independent variables; also this converts partial differential equations with the boundary conditions to ordinary differential equations with appropriate corresponding conditions. In this paper this method is applied to the problem of nonlinear flow of an incompressible second-order fluid past a stretching sheet. Set of five similarity representations are obtained.

Keywords: symmetry analysis, Group Theoretic Method.

Mathematics Subject Classification: 76M60, 34C14.

1. INTRODUCTION

The boundary layer theory emerged at first from the very inspired ideas of Prandtl (1904) when he formulated the boundary-layer equations, in many engineering and industrial applications of fluid flow, one often deals with fluids whose behavior cannot be described. It is well-known that a number of fluids which occur in practical applications, such as molten plastic, polymers, pulps, foods, etc. exhibit a non-Newtonian fluid behavior, the fluids material are complex, the research of boundary layers of non-

Boundary layer theory has been applied successfully to various non-Newtonian fluids models. These non-Newtonian fluids vary greatly among themselves in their physical structure and their responses to the stress, so the difficulty of the study of such fluid problems is to describe the relation between the stress and the strain-rate tensor, the boundary layer theory described by the constitutive equations of this relation for the Newtonian fluid which be linear but for the non-Newtonian. Many constitutive equations of this relation have been proposed by many authors see [7], one of the models of these constitutive equations for the second order fluid formulated by R.S. Rivilin and J.L. Ericksen. The employment of these models in the momentum equations results in highly nonlinear equations of motion. As a result, it is very difficult to solve these equations unless, either there are some simplifying features to the problem or some simplifying assumptions are made.

For two-dimensional flows several authors have employed boundary layer type analysis with different idealized constitutive equations, Ting [24] , Beard and Walters [4] , Sarpkaya and Rainey [21] , Huilgol [11] , Rajagoppal and Gupta [17,18] , Kaloni and Siddiqui [12] and Siddiqui [23] are few to quote, their results are not to accurately applicable for many engineering situations. For example, Hansen and Herzig [8] have noted that in turbulent boundary layers, the two-dimensional boundary layer flows tend to become three-dimensional because of generation of the secondary flows. Thus, for physically acceptable solutions consideration of the three-dimensional boundary layer equations is much more desirable. In an article, Sacheti and Chandran [20] have studied the steady three-dimensional boundary layer flow of a certain kind of second order fluid over a plate. In this article the models of constitutive equations for the second order fluid formulated by el-Saheli [7] is used.

2. Mathematical Formulation

Consider the three-dimensional flow of an incompressible viscoelastic fluid under the usual boundary layer approximations as shown in [7], the flow is governed by the following Equations:

$$\phi_1 : \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0 \tag{2.1}$$

$$\begin{aligned} \phi_2 : & u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - U\frac{\partial U}{\partial x} - W\frac{\partial U}{\partial z} - \frac{\partial^2 u}{\partial z^2} - \bar{\mu}_1\left[u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial y^3} + w\frac{\partial^3 u}{\partial y^2\partial z} + \right. \\ & \left. \frac{\partial u}{\partial y}\left(3\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2 w}{\partial z\partial y}\right) + \frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial x} - 2\frac{\partial w}{\partial z}\right) + 2\frac{\partial w}{\partial y}\left(\frac{\partial^2 u}{\partial z\partial y} + \frac{\partial^2 w}{\partial x\partial y}\right) + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial u}{\partial z} + 2\frac{\partial w}{\partial x}\right)\right] \\ & - \bar{\mu}_2\left[\frac{\partial}{\partial y}\left(\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial y} - 2\frac{\partial u}{\partial y}\frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2\right] = 0 \end{aligned} \tag{2.2}$$

$$\begin{aligned} \phi_3 : & u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - U\frac{\partial W}{\partial x} - W\frac{\partial W}{\partial z} - \frac{\partial^2 w}{\partial y^2} - \bar{\mu}_1\left[u\frac{\partial^3 w}{\partial x\partial y^2} + v\frac{\partial^3 w}{\partial y^3} + w\frac{\partial^3 w}{\partial y^2\partial z} + \right. \\ & \left. \frac{\partial w}{\partial y}\left(3\frac{\partial^2 w}{\partial y\partial z} - \frac{\partial^2 u}{\partial x\partial y}\right) + \frac{\partial^2 u}{\partial y^2}\left(\frac{\partial w}{\partial x} + 2\frac{\partial u}{\partial z}\right) + 2\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial z\partial y} + \frac{\partial^2 w}{\partial x\partial y}\right) + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial z} - 2\frac{\partial u}{\partial x}\right)\right] \\ & - \bar{\mu}_2\left(\frac{\partial}{\partial y}\left(\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{\partial u}{\partial y} - 2\frac{\partial u}{\partial x}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial y}\right)^2\right) = 0 \end{aligned} \tag{2.3}$$

Subject to the boundary conditions:

$$B_1 : u = 0, \text{ at } \omega_1 : y = 0; \quad B_2 : v = 0, \text{ at } \omega_2 : y = 0; \quad B_3 : w = 0, \text{ at } \omega_3 : y = 0$$

$$B_4 : u - U = 0, \quad B_5 : w - W = 0, \quad \text{as} \quad y \to \infty; \tag{2.4}$$

3. Method of Solution and the Group of Transformations

The boundary layer equations are especially interesting from a physical point of view because they have the capacity to admit a large number of invariant solutions. So here the method, for finding similarity solutions of the system of nonlinear BVP's (2.1)-(2.4) is an extension a new approach proposed by Al-Salihi at al [3], which is based upon an improvement of the group theoretical method of Morgan [16] and Moran et al [15], i.e. we will extend the approach form one-parameter group to the two-parameter group.

The approach depends on deduced the conditions under which the group leave the problem invariant not by the usual way of the finite group but using the infinitesimal group methods as alternative way which is simple; and with less steps. In present paper, we shall apply two-parameter transformation groups to the system of Eqs. (2.1)-(2.3) with the boundary conditions (2.4). The system of equations reduces to a system of ordinary differential equation in a single independent variable with appropriate boundary conditions. The procedure is initiated with the group G, a class of two-parameter group of the form

$$G : \overline{S} = C^{s}(a_1, a_2)S + K^{s}(a_1, a_2) \tag{3.1}$$

where S stands for x, y, z, u, v, w, U and W and the $C's$ *and* $K's$ are at least differentiable in each real arguments (a_1, a_2). Thus, in the notation of the above representation, the present nalysis is initiated with a class C_G of two-parameter transformation group in the form

$$G = \begin{cases} s : \begin{cases} \overline{x} = C^{x}(a_1, a_2)x + K^{x}(a_1, a_2) \\ \overline{y} = C^{y}(a_1, a_2)y + K^{y}(a_1, a_2) \\ \overline{z} = C^{z}(a_1, a_2)z + K^{z}(a_1, a_2) \end{cases} \\ \overline{u} = C^{u}(a_1, a_2)u + K^{u}(a_1, a_2) \\ \overline{v} = C^{v}(a_1, a_2)v + K^{v}(a_1, a_2) \\ \overline{w} = C^{w}(a_1, a_2)w + K^{w}(a_1, a_2) \\ \overline{U} = C^{U}(a_1, a_2)U + K^{U}(a_1, a_2) \\ \overline{W} = C^{W}(a_1, a_2)W + K^{W}(a_1, a_2) \end{cases} \tag{3.2}$$

which possesses complete sets of absolute invariants.

Since the importance of the absolute invariants lies in the fact that they become the similarity variables (i.e., the variables of the similarity representations), the initiating an analysis with the group (3.2) assures that absolute invariants (the similarity variables) may be very readily deduced by means of a systematic procedure which proposed by Moran et al [15], and so, the existence of absolute invariants directly from the assumed group and expression of (2.1)-(2.4) in terms of them, be somewhat enough for invariance analysis. As will be shown in the following, if (2.1)-(2.4) are expressible in terms of absolute invariants of a group G, then it will be invariant under G.

In other words, invoking the basic theorem of group theory is also useful for invariance analysis of above case. Since (3.2) can be rewritten in the terms of infinitesimal forms (point translations) with the infinitesimal generators

$$X_1 = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial \mathrm{y}} + \xi_3 \frac{\partial}{\partial \mathrm{z}} + \xi_4 \frac{\partial}{\partial \mathrm{u}} + \xi_5 \frac{\partial}{\partial \mathrm{v}} + \xi_6 \frac{\partial}{\partial \mathrm{w}} + \xi_7 \frac{\partial}{\partial \mathrm{U}} + \xi_8 \frac{\partial}{\partial \mathrm{W}}$$

$$X_2 = \bar{\xi}_1 \frac{\partial}{\partial x} + \bar{\xi}_2 \frac{\partial}{\partial \mathrm{y}} + \bar{\xi}_3 \frac{\partial}{\partial \mathrm{z}} + \bar{\xi}_4 \frac{\partial}{\partial \mathrm{u}} + \bar{\xi}_5 \frac{\partial}{\partial \mathrm{v}} + \bar{\xi}_6 \frac{\partial}{\partial \mathrm{w}} + \bar{\xi}_7 \frac{\partial}{\partial \mathrm{U}} + \bar{\xi}_8 \frac{\partial}{\partial \mathrm{W}}$$

where $\xi's$ and $\bar{\xi}'s$ are the infinitesimals which have valued as following

$$\xi_i = \frac{\partial \bar{S}_i}{\partial a_1}\Big|_{a_1=a_1^0} = \frac{\partial (C^{S_i} S_i + K^{S_i})}{\partial a_1}\Big|_{a_1=a_1^0} = \frac{\partial C^{S_i}}{\partial a_1}\Big|_{a_1=a_1^0} S_i + \frac{\partial K^{S_i}}{\partial a_1}\Big|_{a_1=a_1^0},$$

$$\bar{\xi}_i = \frac{\partial \bar{S}_i}{\partial a_1}\Big|_{a_1=a_1^0} = \frac{\partial (C^{S_i} S_i + K^{S_i})}{\partial a_2}\Big|_{a_2=a_2^0} = \frac{\partial C^{S_i}}{\partial a_2}\Big|_{a_1=a_1^0} S_i + \frac{\partial K^{S_i}}{\partial a_2}\Big|_{a_2=a_2^0},$$

Where S_i stands for $\mathrm{x, y, z, u, v, w, U}$ and $\overline{\mathrm{W}}$, $\mathrm{i} = 1,...,8$

Let $\alpha_{2i-1} = \frac{\partial C^{S_i}}{\partial a_1}\Big|_{a_1=a_1^0}$, $\alpha_{2i} = \frac{\partial K^{S_i}}{\partial a_1}\Big|_{a_1=a_1^0}$ *and* $\beta_{2i-1} = \frac{\partial C^{S_i}}{\partial a_2}\Big|_{a_2=a_2^0}$, $\beta_{2i} = \frac{\partial K^{S_i}}{\partial a_2}\Big|_{a_2=a_2^0}$.

Then $\xi's$, $\bar{\xi}'s$ and infinitesimal generators can be written in the following simple notions

$$\xi_i = \alpha_{2i-1} S_i + \alpha_{2i}, \quad \bar{\xi}_i = \beta_{2i-1} S_i + \beta_{2i}$$

$$X_1 = \sum_i^8 (\alpha_{2i-1} S_i + \alpha_{2i}) \frac{\partial}{\partial S_i} \tag{3.3}$$

$$X_2 = \sum_i^8 (\beta_{2i-1} S_i + \beta_{2i}) \frac{\partial}{\partial S_i} \tag{3.4}$$

Following [5] the point symmetry X_1 and X_2 are admitted by the boundary value problem (2.1)-(2.4) if and only if:

(1) $\mathrm{C_i} X_1^{(k)} \phi_\mu + \bar{\mathrm{C}}_\mathrm{i} X_2^{(k)} \phi_\mu = 0$ when $\phi_\mu = 0$, $\mu = 1,2,3$

(2) $\mathrm{C_i} X_1^{(k-1)} B_j + \bar{\mathrm{C}}_\mathrm{i} X_2^{(k-1)} B_j = 0$ when $B_j = 0$, $\mathrm{j} = 1,2,3,4,5$ on $\omega_\nu = 0$

(3) $\mathrm{C_i} X_1 \omega_\nu + \bar{\mathrm{C}}_\mathrm{i} X_2 \omega_\nu = 0$ when $\omega_\nu = 0$, $\nu = 1,2,3$,

where $X_1^{(k)}$ and $X_2^{(k)}$ are *k*th-extended infinitesimal generators, as discussed in details by Bluman [5], accordingly the necessary and sufficient conditions for boundary value problem (2.1)-(2.4) to be invariant in the form under *k*th-extended group of infinitesimal form (point translations) of (3.2) are

(1) $X_1^{(k)} \phi_\mu = 0$, $X_2^{(k)} \phi_\mu = 0$ when $\phi_\mu = 0$, $\mu = 1,2,3$

(2) $X_1^{(k-1)} B_j = 0$, $X_2^{(k-1)} B_j = 0$ when $B_j = 0$, $\mathrm{j} = 1,2,3,4,5$ on $\omega_\nu = 0$ (3.5)

(3) $X_1 \omega_\nu = 0$, $X_2 \omega_\nu = 0$ when $\omega_\nu = 0$, $\nu = 1,2,3$

because C's are arbitrary constants.

If we can determine (p-2) independent solutions of the system $X_1^{(k)} \phi_\mu = 0$, $X_2^{(k)} \phi_\mu = 0$, $\lambda_1, \lambda_2, ..., \lambda_{p-2}$ (say) where p the number of an independent variables, dependent variables and the derivatives thereof up to the *k*th order, such that λ_1 is functionally independent invariant of the system, satisfying

$$(\alpha_1 x+\alpha_2)\frac{\partial\lambda_1}{\partial x}+(\alpha_3 y+\alpha_4)\frac{\partial\lambda_1}{\partial y}+(\alpha_5 x+\alpha_6)\frac{\partial\lambda_1}{\partial z}=0$$

$$(\beta_1 x+\beta_2)\frac{\partial\lambda_1}{\partial x}+(\beta_3 x+\beta_4)\frac{\partial\lambda_1}{\partial y}+(\beta_5 x+\beta_6)\frac{\partial\lambda_1}{\partial z}=0$$

when the rank of the matrix $\begin{pmatrix}\alpha_1 x+\alpha_2 & \alpha_3 y+\alpha_4 & \alpha_5 z+\alpha_6\\ \beta_1 x+\beta_2 & \beta_3 y+\beta_4 & \beta_5 z+\beta_6\end{pmatrix}$ be two, and

$$\lambda_q=\frac{S_q}{\Gamma_q(x,z)} \tag{3.6}$$

be an invariants of system, satisfying

1. $X_1\lambda_q=0$, $X_2\lambda_q=0$, $\mathrm{q}=2,\ldots,\mathrm{m}+1$ and $S's$ stand for dependent variables.
2. $X_1^{(k)}\lambda_q=0$, $X_2^{(k)}\lambda_q=0$, $\mathrm{q}=(\mathrm{m}+2),\ldots,\mathrm{p}-2$ and $S's$ stand for the derivatives thereof up to the *k*th order.

Then the general form of ϕ_μ, invariant in the form under this group be obtained by equating an arbitrary functions ψ_μ of the (p –2) independent solutions to zero; that is,

$$\psi_\mu(\lambda_1,\lambda_2,\ldots,\lambda_{p-2})=0. \tag{3.7}$$

Since the general form ψ_μ of the ϕ_μ which is invariant under the two-parameter group, ψ_μ can generally be rewritten in terms of its first six arguments, which refer to complete set of an absolute invariants to C_G, (see Morgan [16], Seshadri and Na[22]). This change of the variables is accomplished by a transformation of variables to a complete set of absolute invariants as following:

$$\begin{cases}\eta=\lambda_1\\ g_j=\lambda_{j+1}=F_j(\eta), \quad \mathrm{j}=1,\ldots,5\end{cases} \tag{3.8}$$

i.e., If η is the absolute invariant of the independent variables, then $g_j=F_j(\eta), j=1,\ldots,5$.
This guarantees invariant solution under the group (3.2) Moran et al ([17], page-64), where η, g_1, g_2, g_3, g_4 and g_5 are a complete set of the group. Thus, if change of those variables is made of ψ_μ, from $(\lambda_1,\lambda_2,\ldots,\lambda_{p-2})$ into $(\eta,F_1,\ldots,F_5,\partial F,\ldots,\partial^k F)$ we get

$$\psi_\mu(\lambda_1,\lambda_2,\ldots,\lambda_{p-2})=\overline{\psi}_\mu(\eta,F_1,\ldots,F_5,\partial F,\ldots,\partial^k F)=0 \tag{3.9}$$

Clearly that any element in ψ_μ deferent $(\lambda_1,\lambda_2,\ldots,\lambda_6)$ is also transform into function of new variables. Consequently, In general, we can write the following lemma and theorem in case a class of tow-parameter group G . Consider the situation of n independent variables $\mathrm{x}=(x_1,\ldots,x_n)$ and m dependent variables $\mathrm{u}=(u_1,\ldots,u_m)$.

Lemma 1.1. If a inspection of ϕ_μ has the form ψ_μ under some conditions, then ϕ_μ reduce into function in terms of $(n+m-2)$ absolute invariants under the transformation (3.8). Conversely, if under same transformation (3.8) ϕ_μ reduce into function in terms of $(n+m-2)$ absolute invariants under some conditions, then ϕ_μ has the form ψ_μ.

Proof: This lemma can be proved easily using relations (3.8) and expression (3.9).

Theorem 1.1. If ϕ_μ are expressed in terms of absolute invariants $(n+m-2)$ of G under some conditions, then ϕ_μ is invariant under G.

Proof. Let $z_1,...,z_p$ are the n independent variables $x_1,...,x_n$, the m dependent variables $u_1,...,u_m$ and the derivatives thereof up to the *k*th order; and $\gamma_1,...,\gamma_p$ are infinitesimals and extended infinitesimals of G.

Suppose that ϕ_μ expressed in terms of absolute invariants $(n+m-2)$ under some conditions. Then according to above lemma ϕ_μ has the form ψ_μ i.e.,

$$\phi_\mu(x_1,...,x_n,u_1,...,u_m,\partial u,...,\partial^k u)=\psi_\mu(\lambda_1,\lambda_2,...,\lambda_{p-2})=0$$

So

$$X_1^{(k)}\phi_\mu = X_1^{(k)}\psi_\mu$$

$$=\gamma_1\frac{\partial\psi_\mu}{\partial z_1}+....+\gamma_p\frac{\partial\psi_\mu}{\partial z_p}$$

$$=\gamma_1(\frac{\partial\psi_\mu}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial z_1}+...+\frac{\partial\psi_\mu}{\partial\lambda_{p-2}}\frac{\partial\lambda_{p-2}}{\partial z_1})+....+\gamma_p(\frac{\partial\psi_\mu}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial z_p}+...+\frac{\partial\psi_\mu}{\partial\lambda_{p-2}}\frac{\partial\lambda_{p-2}}{\partial z_p})$$

$$=\frac{\partial\psi_\mu}{\partial\lambda_1}(\gamma_1\frac{\partial\lambda_1}{\partial z_1}+...+\gamma_p\frac{\partial\lambda_1}{\partial z_p})+....+\frac{\partial\psi_\mu}{\partial\lambda_{p-2}}(\gamma_1\frac{\partial\lambda_{p-2}}{\partial z_1}+...+\gamma_p\frac{\partial\lambda_{p-2}}{\partial z_p})$$

$$=\frac{\partial\psi_\mu}{\partial\lambda_1}(0)+....+\frac{\partial\psi_\mu}{\partial\lambda_{p-2}}(0)$$

$$=0$$

similarly $X_2^{(k)}\phi_\mu=0$, and according to the infinitesimal criterion for the invariance, ϕ_μ are invariant under G.

In a similar manner the second condition in (3.5) satisfies if $B's$ are expressible in terms of absolute invariants. Thus, if a third condition in (3.5) is satisfied and $\phi's$ and $B's$ are expressible in terms of absolute invariants, then (3.5) satisfied and BVP's (2.1)-(2.4) be invariant under G.

4. Complete sets of Absolute Invariants

The conditions $X_1\omega_v=0, X_2\omega_v=0,$ where $\omega_v=y=0$ lead to $\alpha_4=\beta_4=0$ i.e., $K^y(a_1,a_2)=0$. Thus, following Moran et al [15], an absolute invariant for independent variables of group G is

$$\eta = y\pi(x,z) \tag{4.1}$$

Where $\pi(x,z)$ will be assigned to following cases; A, $(ax+z+b)^r$, $(x+a)^m(z+b)^r$, $e^{rz}e^{mx}$, $(x+a)^m e^{rz}$, $(z+b)^r e^{mx}$, $(x+a)^m$, $((z+b)^r$, e^{rx} and e^{mx} with A constant. Moreover, for the absolute invariant corresponding to the dependent variables $x, y, z,, v, w, U$ *and* W, following Abd-el-Malek [1, 2] and Hassanien [10], are the following:

$$g_1 = \phi_1\left(\frac{u}{\Gamma_1(x,y)}\right) = F_1,\ g_2 = \phi_2\left(\frac{v}{\Gamma_2(x,y)}\right) = F_2,\ g_3 = \phi_3\left(\frac{w}{\Gamma_3(x,y)}\right) = F_3,$$

$$g_4 = \phi_4\left(\frac{U}{\Gamma_4(x,y)}\right) = F_4,\quad g_5 = \phi_5\left(\frac{W}{\Gamma_5(x,y)}\right) = F_5. \tag{4.2}$$

which correspond with (3.6), without loss of generality, $\phi's$ in (4.2) are selected to be the identity functions. Then we can express $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$, $U(x,z)$ and $W(x,z)$ in terms of the absolute invariants $F_1(\eta)$, $F_2(\eta)$, $F_3(\eta)$, $F_4(\eta)$ and $F_5(\eta)$ in the form

$$u(x,y,z) = \Gamma_1(x,y)F_1,\ v(x,y,z) = \Gamma_2(x,y)F_2,\ w(x,y,z) = \Gamma_3(x,y)F_3,$$

$$U(x,y,z) = \Gamma_4(x,y)F_4,\ W(x,y,z) = \Gamma_5(x,y)F_5. \tag{4.3}$$

Since $\Gamma_4(x,y)$, $\Gamma_5(x,y)$, $U(x,y)$ and $W(x,y)$ are independent of y, whereas η depends on y, it follows that F_4 and F_5 in (4.3) must be equal to a constant. Therefore

$$\begin{aligned} &u(x,y,z) = \Gamma_1(x,y)F_1, \\ &v(x,y,z) = \Gamma_2(x,y)F_2, \\ &w(x,y,z) = \Gamma_3(x,y)F_3, \\ &U(x,y,z) = \Gamma_4(x,y)U_0 \quad and \\ &W(x,y,z) = \Gamma_5(x,y)W_0. \end{aligned} \tag{4.4}$$

where U_0 and W_0 are arbitrary constants.

5. The Invariance Analysis and Reductions to ODEs.

As the general analysis proceeds, the established forms of the dependent and independent absolute invariants are used to obtain ordinary differential equations. The invariance in the form of $\phi's$ and $B's$ take place if they are expressed in terms of absolute invariants see [3], first we will start with the auxiliary conditions, i.e., (we will examine whether the auxiliary conditions are appropriate to (4.1) and (4.4) or not, because these auxiliary conditions must transform properly, otherwise no similarity solution is possible), substituting (4.1) and (4.4) into (2.4) we get

$$F_1(0) = 0, F_1(0) = 0, F_1(0) = 0, \quad at \quad \eta = 0 \qquad \text{and}$$

$$F_1(\eta) = U_0\frac{\Gamma_4(x,z)}{\Gamma_1(x,z)},\ F_3(\eta) = W_0\frac{\Gamma_5(x,z)}{\Gamma_3(x,z)} \qquad \text{as} \qquad \eta \to \infty$$

which be appropriate with absolute invariants if $\frac{\Gamma_4(x,z)}{\Gamma_1(x,z)} = a$ and $\frac{\Gamma_5(x,z)}{\Gamma_3(x,z)} = b$ where the a and b are arbitrary constants, for easily we can take a = b = 1. The boundary conditions become in the following form:

$$F_1(0) = 0, F_1(0) = 0, F_1(0) = 0, \quad at \quad \eta = 0 \qquad and$$
$$F_1(\eta) = U_0,\ F_3(\eta) = W_0 \qquad \text{as} \qquad \eta \to \infty \tag{5.1}$$

Put $\Gamma_4(x,z) = \Gamma_1(x,z)$ and $\Gamma_5(x,z) = \Gamma_3(x,z)$ into (4.1) and (4.4) and then substitute into Eqs. (2.1)-(2.3), yields, after dividing the resulting three equations by $\pi(x,z)\Gamma_2(x,z)$, $\pi^3(x,z)\Gamma_2(x,z)\Gamma_1(x,z)$ and $\pi^3(x,z)\Gamma_2(x,z)\Gamma_3(x,z)$ respectively, and rearranging the terms,

$$F_2' + C_4F_1 + C_2F_3 + C_3\eta F_1' + C_1\eta F_3' = 0 \tag{5.2}$$

$$\begin{aligned}
&-\bar{\mu}_1F_2F_1''' + C_4C_5(F_1^2 - 1) + C_5C_6(F_1F_3 - 1) + C_3C_5\eta F_1F_1' - C_9F_1'' \\
&+ C_5F_2F_1' + C_1C_5\eta F_3F_1' - C_6[\bar{\mu}_1F_3F_1'' + (\bar{\mu}_2 + \bar{\mu}_1)(F_1F_3'' + 2F_1'F_3')] \\
&- C_4[(F_1')^2(2\bar{\mu}_2 + 3\bar{\mu}_1) + 2\bar{\mu}_1F_1F_1''] + C_2[(\bar{\mu}_2 + \bar{\mu}_1)(F_1'F_3' + 2F_3F_1'')] \\
&- C_7(\bar{\mu}_2 + 2\bar{\mu}_1)[(F_3')^2 + F_3F_3''] - C_8(\bar{\mu}_2 + 2\bar{\mu}_1)[(F_3')^2 + 2\eta F_3'F_3''] \\
&- C_3[(2\bar{\mu}_2 + 3\bar{\mu}_1)(F_1')^2 + 2\bar{\mu}_1F_1F_1'' + 2\eta(\bar{\mu}_2 + 2\bar{\mu}_1)F_1'F_1'' + \eta\bar{\mu}_1F_1F_1'''] \\
&- C_1[(\bar{\mu}_2 + \bar{\mu}_1)F_1'F_3' + 2\bar{\mu}_1F_3F_1'' + \eta\bar{\mu}_1F_3F_1'''] = 0
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
&-\bar{\mu}_1F_2F_3''' + C_1C_5\eta[F_3F_3'] + C_2C_5[F_3^2 - 1] + C_3C_5[\eta F_1F_3'] + C_5F_2F_3' \\
&+ C_{10}C_5(F_1F_3 - 1) - C_{10}[(\bar{\mu}_2 + \bar{\mu}_1)(2F_1'F_3' + F_3F_1'') + \bar{\mu}_1F_1F_3''] \\
&- C_{11}(\bar{\mu}_2 + 2\bar{\mu}_1)[(F_1')^2 + 2\eta F_1'F_1''] - C_{12}(\bar{\mu}_2 + 2\bar{\mu}_1)[(F_1')^2 + F_1F_1''] \\
&+ C_4(\bar{\mu}_2 + \bar{\mu}_1)[(2F_1F_3'' + F_1'F_3')] - C_2[(2\bar{\mu}_2 + 3\bar{\mu}_1)(F_3')^2 + 2\bar{\mu}_1F_3F_3''] \\
&- C_3[(\bar{\mu}_2 + \bar{\mu}_1)F_1'F_3' + 2\bar{\mu}_1F_1F_3'' + \eta\bar{\mu}_1F_1F_3'''] - C_9F_3'' \\
&- C_1[(2\bar{\mu}_2 + 3\bar{\mu}_1)(F_3')^2 + 2\eta(\bar{\mu}_2 + 2\bar{\mu}_1)F_3'F_3'' + 2\eta\bar{\mu}_2F_3F_3'' + \eta\bar{\mu}_1F_3F_3'''] = 0
\end{aligned} \tag{5.4}$$

where a primes denotes differentiation with respect to η; and $C_1,...,C_{12}$ are as following:

$$C_1 = \frac{\Gamma_3}{\pi^2\Gamma_2}\frac{\partial\pi}{\partial z},\ C_2 = \frac{1}{\pi\Gamma_2}\frac{\partial\Gamma_3}{\partial z},\ C_3 = \frac{\Gamma_1}{\pi^2\Gamma_2}\frac{\partial\pi}{\partial x},\ C_4 = \frac{1}{\pi\Gamma_2}\frac{\partial\Gamma_1}{\partial x},\ C_5 = \frac{1}{\pi^2},$$
$$C_6 = \frac{\Gamma_3}{\pi\Gamma_2\Gamma_1}\frac{\partial\Gamma_1}{\partial z},\ C_7 = \frac{\Gamma_3}{\pi\Gamma_2\Gamma_1}\frac{\partial\Gamma_3}{\partial x},\ C_8 = \frac{\Gamma_3^2}{\pi^2\Gamma_2\Gamma_1}\frac{\partial\pi}{\partial x},\ C_9 = \frac{1}{\pi\Gamma_2},$$
$$C_{10} = \frac{\Gamma_1}{\pi\Gamma_2\Gamma_3}\frac{\partial\Gamma_3}{\partial x},\ C_{11} = \frac{\Gamma_1^2}{\pi^2\Gamma_2\Gamma_3}\frac{\partial\pi}{\partial z}\ and\ C_{12} = \frac{\Gamma_1}{\pi\Gamma_2\Gamma_3}\frac{\partial\Gamma_1}{\partial z} \tag{5.5}$$

Inasmuch as the first term of (5.2)-(5.4) has the constant coefficients. So to reducing to an expression in the single independent variable η, it is necessary that the remaining coefficients be constants or functions of η alone. Thus, since π and $\Gamma's$ are independent of y, $C's$ are constants; and to be determined for each individual case corresponding to each set of absolute invariants. The equations $C_5 = \frac{1}{\pi^2}$ and $C_9 = \frac{1}{\pi\Gamma_2}$; state that π and Γ_2 are constants so the absolute invariant η take only one case when $\eta = \text{Constant}$. By considering C_5 and C_9 may be taken to be unity, it follows that $\pi = \Gamma_2 = 1$ and $C_1 = C_3 = C_8 = C_{11} = 0$, and by considering $F_1(\eta) = F'(\eta)$ and $F_3(\eta) = G'(\eta)$ and rearranging the terms, we can rewrite the above equation in other form, Thus according to (5.2) $F_2(\eta) = -C_4F(\eta) - C_2G(\eta)$ and we get

$$\begin{aligned}&\overline{\mu}_1[C_2(F^{iv}G + F''G'' + 2G'F''') - C_4\left(3(F'')^2 + 2F'F''' - F^{iv}F\right)\\&- C_6(2F''G'' + F'G''' + G'F''') - 2C_7((G'')^2 + G'G''')] - C_2GF'' - F'''\\&- \overline{\mu}_2[2C_4(F'')^2 - C_2(F''G'' + 2G'F''') + C_6(2F''G'' + F'G''')\\&+ C_7((G'')^2 + G'G''')] + C_4((F')^2 - FF'' - 1) + C_6(F'G' - 1) = 0\end{aligned} \tag{5.6}$$

$$\begin{aligned}&\overline{\mu}_1[C_4\left(2F'G'' + F''G'' + FG^{iv}\right) + C_2(3(G'')^2 + 2G'G''' - GG^{iv})\\&+ C_{10}(2F''G'' + F'''G' + G'''F') + 2C_{12}((F'')^2 + F'F''')] - C_4G''F - G'''\\&- \overline{\mu}_2[2C_2(G'')^2 - C_4(2F'G''' + F''G'') + C_{10}(2F''G'' + G'F''')\\&+ C_{12}((F'')^2 + F'F''')] + C_2[(G')^2 - G''G - 1] + C_{10}(F'G' - 1) = 0\end{aligned} \tag{5.7}$$

with the boundary conditions (5.1).

Now to evaluate the $C's$ appearing in the ordinary differential equations (5.6) and (5.7) and consequently to evaluate the corresponding expressions of the functions Γ_1 and Γ_3 we get

$\Gamma_1 = C_4x + \frac{r}{C_7}z + k_1$ and $\Gamma_3 = C_2z + \frac{r}{C_6}x + k_2$ with $C_2 = C_6$, $C_4 = C_{10} = \frac{r^2}{C_2^2C_7}$, $C_{12} = \frac{r^2}{C_2C_7^2}$ and $\frac{K_1}{K_2} = \frac{r}{C_2C_7} = \frac{C_4C_2}{r} = \frac{C_{12}C_7}{r}$. Where K_1 and K_2 are an integrating constants and r arbitrary constant. That was the solution for case when the Γ_1 and Γ_3 are functions for both x, z sub-cases are appear when they are functions for only one variable,

5.1. Solution when $\Gamma_1 = \Gamma_1(x)$ *and* $\Gamma_3 = \Gamma_3(x)$

For this sub-case with (5.5) we have,

$$\Gamma_1 = C_4x + K_1,\ \ \Gamma_3 = K_2(C_4x + K_1)^{\frac{C_{10}}{C_4}},\ \ C_2 = C_6 = C_{12} = 0$$

where $C_4 = C_{10}, \frac{C_7}{C_{10}} = K_2$. By substituting the values of constants obtained above into Eqs. (5.6) and (5.7), we get

$$\begin{aligned}&\overline{\mu}_1[C_4\left(F^{iv}F - 3(F'')^2 - 2F'F'''\right) - 2C_7((G'')^2 + G'G''')] - F'''\\&+ \overline{\mu}_2[2C_4(F'')^2 + C_7((G'')^2 + G'G'')] + C_4((F')^2 - FF'' - 1) = 0\end{aligned} \tag{5.8}$$

$$\bar{\mu}_1[C_4\left(2F'G''' + F''G'' + FG^{iv}\right) - C_{10}(2F''G'' + F'''G' - G'''F')] - C_4G''F - G'''$$
$$- \bar{\mu}_2[C_{10}(2F''G'' + G'F''') - C_4(2F'G''' + F''G'')] + C_{10}(F'G' - 1) = 0 \qquad (5.9)$$

with the boundary conditions (5.1)

5.2. Solution when $\Gamma_1 = \Gamma_1(z)$ *and* $\Gamma_3 = \Gamma_3(z)$

For this sub-case with (5.5) we have,

$$\Gamma_1 = K_2(C_2 z + K_1)^{\frac{C_6}{C_2}},\ \Gamma_3 = C_2 z + K_1,\ C_4 = C_7 = C_{10} = 0$$

where $C_4 = C_{10}, \dfrac{C_7}{C_{10}} = K_2$. By substituting the values of constants obtained above into Eqs. (5.6) and (5.7), we get:

$$\bar{\mu}_1[C_2(F^{iv}G + F''G'' + 2G'F''') - C_6(2F''G'' + F'G''' + G'F''')] - C_2GF'' - F'''$$
$$- \bar{\mu}_2[C_6(2F''G'' + F'G''') - C_2(F''G'' + 2G'F''')] + C_6(F'G' - 1) = 0 \qquad (5.10)$$

$$\bar{\mu}_1[C_2(GG^{iv} - 3(G'')^2 - 2G'G''') - 2C_{12}((F'')^2 + F'F''')] - G'''$$
$$- \bar{\mu}_2[2C_2(G'')^2 + C_{12}((F'')^2 + F'F''')] + C_2[(G')^2 - G''G - 1] = 0 \qquad (5.11)$$

with the boundary conditions (5.1)

5.3. Solution when $\Gamma_1 = \Gamma_1(x)$ and $\Gamma_3 = \Gamma_3(z)$

For this sub-case with (5.5) we have,

$$\Gamma_1 = (C_4 z + K_1),\ \Gamma_3 = C_2 z + K_2,\ C_6 = C_7 = C_{10} = C_{12} = 0.$$

By substituting the values of constants obtained above into Eqs. (5.6) and (5.7), we get:

$$\bar{\mu}_1[C_4\left(F^{iv}F - 3(F'')^2 - 2F'F'''\right) + C_2(F^{iv}G + F''G'' + 2G'F''')] - C_2GF'' - F'''$$
$$- \bar{\mu}_2[2C_4\left(F''\right)^2 - C_2(F''G'' + 2G'F''')] + C_4((F')^2 - FF'' - 1) = 0 \qquad (5.12)$$

$$\bar{\mu}_1[C_4\left(2F'G''' + F''G'' + FG^{iv}\right) + C_2(GG^{iv} - 3(G'')^2 - 2G'G''')] - C_4G''F - G'''$$
$$- \bar{\mu}_2[2C_2(G'')^2 - C_4(2F'G''' + F''G'')] + C_2[(G')^2 - G''G - 1] = 0 \qquad (5.13)$$

with the boundary conditions (5.1)

5.4. Solution when $\Gamma_1 = A$ and $\Gamma_3 = B$, where A, B are constants.

For this sub-case with (5.5) we get, $C_2 = C_4 = C_6 = C_7 = C_{10} = C_{12} = 0$. By substituting the values of constants obtained above into Eqs. (5.6) and (5.7), we get:

$$F''' = 0 \qquad (5.14)$$
$$G''' = 0 \qquad (5.15)$$

with the boundary conditions (5.1)

ACKNOWLEDGMENT

AKA (Permenant Address: Department of Mathematics, Al-Baida University, Al-Baida, Republic of Yemen) acknowledges with thanks the financial support received from Al-Baida University and Government of Republic of Yemen. He also thanks to Indian Council for Culture and Relations (ICCR) for providing Scholarship to pursue research in India.

AKA and AHH express thanks to UGC (India) for UGC-SAP-DRS (Phase-II) provided to the Department of Mathematics, Sardar Patel University, Valabh Vidyanagar under which the work was carried out.

REFERENCES

[1] Abd-el-Malek M. B. and Badran N.A. : Group method analysis of steady free-convective laminar boundary-layer flow on a nonisothermal vertical circular cylinder, J. Comput. Appl. Math. 36, (1991), 227-238.

[2] Abd-el-Malek M. B., Boutros Y.Z. and Badran N.A. : Group method analysis of unsteady free-convective boundary-layer flow on a nonisothermal vertical at plate, J. Eng. Math. 24, (1990), 343-368.

[3] Al-Salihi A. K., Hasmani A. H. and Timol M. G. : The New Systematic Procedure In Group Theoretic Methods with Applications to Class of Boundary Layer Flows, IJAMAS, (2010), to appear.

[4] Beard D. W. and Walters K. : Elastico-viscous boundary-layer flows, Proc. Camb. Philos. Soc., 60, (1964), 667-674.

[5] Bluman G. W. and Anco S. C. : Symmetry and Integration Methods for Differential Equations, Springer-Verlag, New York, 2002.

[6] Cohen A. : An Introduction to the Lie Theory of One-parameter Groups. Heath, Boston, 1911.

[7] El-Saheli A. : Some Applications of Similarity Solutions in Fluid-Flow Problems, Ms.c Dissertation, University of Windsor, Windsor, Ontario, Canada, 1971.

[8] Hansen A. G. and Herzig H. Z. : On possible similarity solutions for three-dimensional, incompressible, laminar boundary layers NASA Tech. Notes, pp. 3768, 1956.

[9] Hansen A. G. : Similarity Analyses of Boundary Value Problems in Engineering, Prentice-Hall, New Jersey, 1964.

[10] Hassanien I. A., Salama A. A. and Hosham H. A. : Group theoretic method analysis for unsteady Boundary layer flow near a stagnation point, TAIWANESE JOURNAL OF MATH-EMATICS. 9(4), (2005), 639-660.

[11] Huilgol R. R. : on uniqueness and nonuniqueness in the plane creeping flows of second order fluids, SUML Appl. Math., 24, (1973), 226-233.

[12] Kaloni P. N. and Siddiqui A.M. : The flow of a second grade fluid, Int. J. Engng. Sei., 21(10), (1983), 1157−1169.

[13] Moran M. J. and Marshek K. M. : A tabular approach to similarity analyses, Mech E and UMIST, 3(2), (1975), 127-136.

[14] Moran M. J. and Gaggioli R. A. : Reduction of the Number of Variables in Systems of Partial Differential Equations with Auxiliary Conditions, SIAM J. Appl. Math., 16, (1968), 202-215.

[15] Moran M. J. Gaggioli R. A. and Scholten W. B. : A New Systematic Formalism for Similarity Analysis, with Applications to Boundary Layers Flows, Tech. Summary Rept., 918, Mathematics Research Center, University of Wisconsin, Madison, 1968.

[16] Morgan A. J. A. : the Reduction by One of the Number of Independent Variables in Some Systems of Partial Differential Equations, Quart. J. Math. 2, (1952), 250-259.

[17] Rajagopal K. R. and Gupta A.S. : Flow and stability of second grade fluid between two paralle1 plates rotating about non coincident axes, Int. J. Engng. Sci., 19(11), (1981) 1401-1409.

[18] Rajagopal K.R and Gupta A.S. : On a class of exact solutions to the equations of motion of a second grade fluid, Int J. Engng. Sci 19, (1981), 1009-1014.

[19] Reiner M. and Abir D. eds., Second-order Effects in Elasticity, Plasticity, And Fluid Dynamics, Pergamon Press, New York, 1964.

[20] Sacheti N. C. and Chandran P. : On similarity solutions for three-dimensional flow of second order fluids, J. Phy. Soc. Japan, 66(3), (1997), 618-622.

[21] Sarpkaya T. and Rainey P.G. : Stagnation point flow of a second-order viscoeiastic fluid, Acta Mech. 11, (1971), 237-241.

[22] Seshadri R. S. and Na T.Y. : Group Invariance in Engineering Boundary Value Problems, Springer-Verlag, New York, 1985.

[23] Siddiqui A. M. : Flow problems of the Fluids of Differential Type, Ph.D. Thesis, Uni. Of Windsor, Windsor, Ont., 1986.

[24] Ting T. W. : Certain non-steady flows of second-order fluids, Arch Rat. Mech Anal., 14,(1963), 1-26.