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Edge Removal and Addition with Variants of Domination

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Abstract

In this paper we study the concept of edge critical graph w.r.t variants of Domination like total domination and K-domination. We have derived characterization of edges which are responsible to increase or decrease total domination number and K-domination number of a graph. We have also given examples to illustrate the conditions given in theorems.

Keywords: Dominating set,Domination number,Total domination, K-domination.

AMS subject classification (2000): 05c69

1. INTRODUCTION:

Suppose G is a graph. S is a Υ set. If we remove an edge from graph or if we add an edge between two non-adjacent vertices then domination number increase or decrease or remains unchanged. This topic has been studied by several authors and reader may refer to references. We shall prove results regarding this phenomenom for total domination and K-domination.

2. **Definitions of K domination:**

2.1 K-dominating Set:

$\rightarrow$ A set $S \subseteq V$ of vertices in a graph G=(V,E) is a K-dominating set if for every vertex $u \in V \setminus S$, $|N(u) \cap S| \geq K$.

2.2 Minimal K-dominating Set:

$\rightarrow$ A K-dominating Set S is said to be a minimal K-dominting set if no proper subset $S' \subseteq S$ is a dominating Set.

2.3 Minimum K-dominating Set:

$\rightarrow$ A minimal K-dominating set with minimum cardinality is called minimum K-dominating Set.

2.4 K-domination number:

$\rightarrow$The K domination number $\Upsilon_k(G)$ of a graph G equals minimum cardinality of a K-dominating Set in G..

2.5 $\Rightarrow$It is useful to partition the Edges of G into two sets according to how their removal affects $\Upsilon_k(G)$.

Let $E = E_K^{+} \cup E_K^{O}$ for

$E_K^{+} = \{e \in E(G) / \Upsilon_K(G-e) > \Upsilon_k(G)\}$

$E_K^{o} = \{e \in E(G) / \Upsilon_K(G-e) = \Upsilon_k(G)\}$

3 **Definitions of Total domination:**

3.1 Total dominating Set:

A set $S \subseteq V$ of vertices in a graph G=(V,E) is a total dominating set if for every vertex $v \in V$ there exist a vertex $u \in S$ such that u is adjacent to v.

3.2 Minimal Total dominating Set:

$\rightarrow$A Total dominating Set S is said to be a minimal Total dominating set if no proper subset $S' \subseteq S$ is a Total dominating Set.

3.3 Minimum Total dominating Set:

$\rightarrow$A minimal Total dominating set with minimum cardinality is called minimum Total dominating Set.

3.4 Total domination number:

$\rightarrow$The Total domination number $\Upsilon_t(G)$ of a graph G equals minimum cardinality of a Total dominating Set in G..

3.5 $\Rightarrow$It is useful to partition the Edges of G into two sets according to how their removal affects $\Upsilon_t(G)$.

Let $E = E_t^{+} \cup E_t^{O}$ for

$E_t^{+} = \{e \in E(G) / \Upsilon_t(G-e) > \Upsilon_t(G)\}$

$E_t^{o} = \{e \in E(G) / \Upsilon_t(G-e) = \Upsilon_t(G)\}$

4 ASSUMPTION:

(1) In this paer, we have assumed that K is any integer $K \geq 1$.
(2) All graphs considered are simple
(3) Totally dominating set contains atleast two vertices.
(4) If u and v are end vertices of any edge e, than e will also be denoted by uv.

5 . PART 1 EDGE REMOVAL/DELETION:-------------------

First we consider the case of removing an edge from given graph. It may be noted that in this case K-domination number and Total domination number will not decrease.

5.1 *Theorem 1*:

An edge e= $P_1P_2 \in Ek^+ \Leftrightarrow$ (for every Υ_K Set S)

$P_1 \in S$, $P_2 \notin S$ & P_2 is adjacent to exactly K vertices of S including P_1.

Proof:

$\Leftarrow$: sufficiency

Let S be a Υ_K Set.

Let e = P_1P_2. $P_1 \in S$, $P_2 \notin S$ & P_2 is adjacent to exactly K vertices of S including P_1.

So if we remove e, then P_2 will be adjacent to exactly K-1 vertices of S.

So, S will not be K-dominating Set in $G \setminus \{e\}$.

This is true for every Υ_K Set S of G.

There for $\Upsilon_K(G-e) > \Upsilon_k(G)$

Necessity:

$\Rightarrow$ Let e $\in Ek^+$

Means removal of e increase K domination number.

- Let P_1 & $P_2 \in S$ or P_1 & $P_2 \notin S$

  then S is K dominating Set in $G \setminus \{e\}$

  Hence $\Upsilon_K(G \setminus e) \le \Upsilon_K(G)$

.

- Let $P_1 \in S$ & $P_2 \notin S$

  Suppose P_2 is adjacent to atleast K+1 vertices of S including P_1, then removing e, still P_2 will be adjacent to K vertices &

  hence e $\notin Ek^+$

  Which is contradiction.

- Let $P_1 \in S$ & $P_2 \notin S$

  Suppose P_2 is adjacent to atmost K-1 vertices of S including P_1,

  Then S is not K dominating Set.

  Which is a contradiction.

So, e = P_1P_2 then $P_1 \in S$, $P_2 \notin S$ & P_2 is adjacent to exactly K vertices of S including P_1. □

5.2 **<u>*Theorem 2*:</u>**

An edge e = $P_1P_2 \in E_K^{\circ}$

$\Leftrightarrow$ there exist a Υ_K set S such that

(1) P_1 and $P_2 \notin S$ or P_1 and $P_2 \in S$

(2) $P_1 \in S$, $P_2 \notin S$ & P_2 is adjacent to atleast K+1 vertices of S.

<u>Proof:</u>

$\Leftarrow$ <u>sufficiency</u>

Let (1) P_1 and $P_2 \notin S$ or P_1 and $P_2 \in S$

Let T be a Υ_K Set in $G \setminus \{e\}$.

Then T is also a K dominating Set in G with |T|<|S|

Which is a Contradiction.

Therefore S is a minimum K dominating Set in $G \setminus \{e\}$

So, $\Upsilon_K(G \setminus e) = |S| = \Upsilon_K(G)$ means e $\in E_K^{\circ}$

Let (2) $P_1 \in S$ & $P_2 \notin S$ is adjacent to atleast K+1 vertices of S.

Then if we remove e then still $P_1 \in S$ and P_2 will be adjacent to K vertices of S.

So, S is K dominating Set in G.

Let T be Υ_K Set in $G \setminus \{e\}$ with |T|<|S|

Now, P_2 is adjacent to atleast K vertices of T.

So, P_2 is adjacent to atleast K+1 vertices of S.

Now, if we add e = P_1P_2 then T is Υ_K set in G with |T|<|S|.

which is contradiction.

So, S is Υ_K Set in $G \setminus \{e\}$. So, e = $P_1P_2 \in E_K^{\circ}$

$\Rightarrow$ <u>:</u> **<u>Necessity:</u>**

Let e = $P_1P_2 \in E_K^{\circ}$

means removal of e does not affect K domination number.

Let S be a Υ_K Set of $G \setminus \{e\}$. then |S|= Υ_K (G)

(1) P_1 and $P_2 \notin S$ or P_1 and $P_2 \in S$

then theorem is proved.

(2) Let $P_1 \in S$ and $P_2 \notin S$.

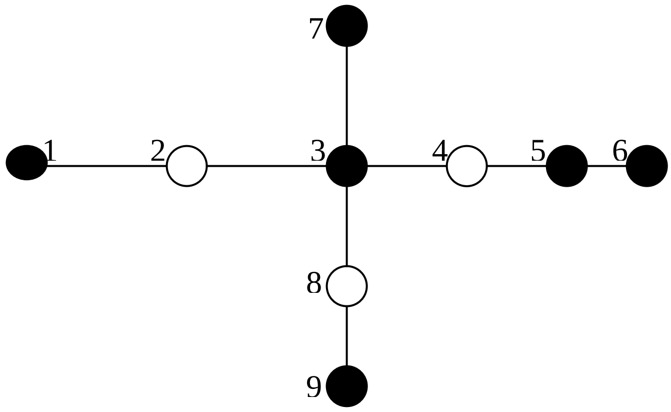
Now P_2 is adjacent to atleast K vertices of S in $G \setminus \{e\}$.

Hence P_2 is adjacent to atleast K+1 vertices including P_1 of S in G.

Thus S is a Υ_K Set of G,

Which satisfies required conditions. □

5.3 **Example: 1** Here we give an example of a graph which represents 2 dominating set and effect of edge removal.



In above Graph G, Υ_2 set S={1,3,5,6,7,9} |S|=6

E_2^+ ={12,23,45,38,89}

E_2^0 ={37,34,56}

5.4 ***Theorem 3*:**

An edge e = $P_1P_2 \in E_t^+ \Leftrightarrow$ For every Υt Set S {e $\notin Et^i$ where Et^i =(e$\in E(G)/\, G-e$ has an isolate vertex)}

one of following two conditions holds.

(1) $P_1 \in S$ & $P_2 \notin S$ with $N(P_2) \cap S = \{P_1\}$

(2) $P_1 \& P_2 \in S \Rightarrow$ $N(P_2) \cap S = \{P_1\}$ or $N(P_1) \cap s = \{P_2\}$

Proof:

$\Leftarrow$: sufficiency

Let S be a Υ_t Set of G.

Suppose (1) holds,

then after removing e from G, P_2 will be vertex which is not adjacent to any vertex of S.

Thus S is not a total dominating Set in $G\setminus\{e\}$

Suppose (2) holds,

then after removing e from G , P_1 or P_2 is not adjacent to any vertex of S.

Thus S is not total dominating Set in $G\setminus\{e\}$

From both the cases above, it follows that S is not total dominating Set in $G\setminus\{e\}$

This is true for every Υ_t Set S of G.

Thus $\Upsilon_t(G \setminus e) > \Upsilon_t(G)$

So, e$\in E_t^+$.

$\Rightarrow$ **Necessity:**

Let S is a Υ_t Set.

e = $P_1P_2 \in E_t^+$.

case -1:

Let P_1 & P_2 $\notin$S

Then S is a totally dominating Set in $G \setminus \{e\}$.

$\Upsilon_t(G \setminus e) \le \Upsilon_t(G)$

So atleast one vertex should be $\in$S.

Case-2:

Let $P_1 \in$S & $P_2 \notin$S.

Since $\Upsilon_t(G \setminus e) > \Upsilon_t(G)$. S can not be totally dominating Set in $G \setminus \{e\}$

So, P_2 is not adjacent to any vertex of S in $G \setminus \{e\}$.

So, N(P_2) $\cap$S = { P_1 }

Case-3:

Let P_1 & P_2 $\in$S , again Since $\Upsilon_t(G \setminus e) > \Upsilon_t(G)$, S can not be totally dominating Set in $G \setminus \{e\}$.

If P_1 is not adjacent to any vertex of S in $G \setminus \{e\}$

then N(P_1) $\cap$ S = { P_2 }

If P_2 is not adjacent to any vertex of S in $G \setminus \{e\}$

then N(P_2) $\cap$ S = { P_1 }.

5.5 *Theorem 4*:

An edge e = $P_1P_2 \in Et^o$ $\Leftrightarrow$ If there exist a Υ_t Set S in G such that one of following holds.

{e $\notin$ Et^i where Et^i =(e$\in E(G) / G - e$ has an isolate vertex)}

(1) P_1 & P_2 $\notin$ S

(2) P_1 $\in$ S & P_2 $\notin$ S and N(P_2) $\cap$S contains a vertex different from P_1.

(3) P_1 & $P_2 \in$S $\Rightarrow$ N(P_1) $\cap$S contains atleast two vertices.

and N(P_2) $\cap$S contains atleast two vertices.

Proof:

$\Leftarrow$: sufficiency

Let S be Υ_t Set in G

Let (1) P_1 and $P_2 \notin S$

So S is totally dominating Set in $G \setminus \{e\}$

Let T be a Υ_t Set in $G \setminus \{e\}$ with |T|<|S|

.Then T is also a totally dominating Set in G with |T|<|S|
Which is a Contradiction.

Therefore S is a minimum totally dominating Set in $G \setminus \{e\}$

So, $\Upsilon_t(G \setminus e) = |S| = \Upsilon_t(G)$ means e $\in E_t \circ$

Let (2) $P_1 \in S$ & $P_2 \notin S$ and $N(P_2) \cap S$ = contains a vertex different from P_1.

So S is totally dominating Set in $G \setminus \{e\}$

Let T be a Υ_t Set in $G \setminus \{e\}$ with |T|<|S|

.

Then T is also a totally dominating Set in G with |T|<|S|
Which is a Contradiction.

Therefore S is a minimum totally dominating Set in $G \setminus \{e\}$

So, $\Upsilon_t(G \setminus e) = |S| = \Upsilon_t(G)$ means e $\in E_t \circ$

Let (3) P_1 & $P_2 \in S \Rightarrow$ $N(P_1) \cap S$ contains atleast two vertices.

and $N(P_2) \cap S$ contains atleast two vertices.

So S is totally dominating Set in $G \setminus \{e\}$

Let T be a Υ_t Set in $G \setminus \{e\}$ with |T|<|S|

.

Then T is also a totally dominating Set in G with |T|<|S|
Which is a Contradiction.

Therefore S is a minimum totally dominating Set in $G \setminus \{e\}$

$e = P_1P_2 \in Et^o$

Necessity:

$\Rightarrow$ Let e = $P_1P_2 \in E_t \circ$

means removal of e does not affect total domination number.

Let S be a Υ_t Set of $G \setminus \{e\}$. then |S|= Υ_t (G)

(1) P_1 and $P_2 \notin S$

then theorem is proved.

(2) $P_1 \in S$ & $P_2 \notin S$

Now P_2 is adjacent to a vertex Q of S in $G \setminus \{e\}$.

Hence P_2 is adjacent to Q of S including P_1 of S in G.

Thus S is a Υ_t Set of G,

Which satisfies required conditions.

(3) P_1 & $P_2 \in S$

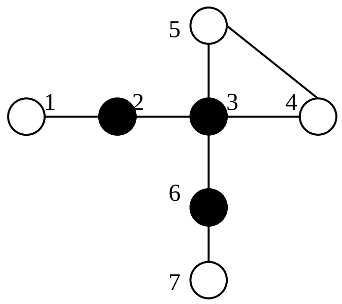
Now P_2 is adjacent to a vertex of S in $G \setminus \{e\}$

Hence N(P_2) $\cap$S contains atleast two vertices in G.

Similarly P_1 is adjacent to a vertex of S in $G \setminus \{e\}$

Hence N(P_1) $\cap$S contains atleast two vertices in G. □

5.6 <u>Example: 2</u> Here we give an example of a graph which represents total dominating set and effect of edge removal between two adjacent vertices.



Here S={2,3,6} is a Υt Set. |S|=3 E_t^+ ={23,34,35,36} & Et^o ={45} & Et^i ={12,67}

6. <u>PART 2 EDGE ADDITION:====================</u>

Now we consider the case of adding an edge between two non-adjacent vertices of given graph. It may be noted that in this case K-domination number & Total domination number will not increase.

6.1 <u>Remark:</u>

It may be noted that if u and v are two non-adjacent vertices of graph G. and if $\Upsilon_k(G+uv) < \Upsilon_k(G)$ then $\Upsilon_k(G+uv) = \Upsilon_k(G) - 1$.

6.2 <u>*Theorem 5*:</u>

Let u and v be two non-adjacent vertices of G and e=uv then $\Upsilon_k(G+e) < \Upsilon_k(G)$

$\Leftrightarrow$

There is a subset S of V(G) such that |S|< $\Upsilon_k(G)$ and following two conditions are satisfied.

(1) $u \in S \; and \; v \notin S$ & v is adjacent to exactly k-1 vertices of S in G

(2) S is a k dominating set in the subgraph G\\$\{v\}$

<u>Proof:</u> <u>sufficiency</u>

$\Leftarrow$

Let S be subset of V(G) such that |S|< $\Upsilon_k(G)$ both conditions holds.

Then S is a K dominating set in G\$\{v\}$ but not in G.

So, s+ $\{v\}$ will be K dominating set in G. (by 2)

If we add e=uv then S will be K dominating set in G+e and
|S|< $\Upsilon_k(G)$. So, $\Upsilon_k(G+e) < \Upsilon_k(G)$

⇒ **necessity**

Let S be a Υ_k set in $G+\{e\}$ *then* $|s| < \Upsilon_k(G)$

Let u and v be two non-adjacent vertices and $\Upsilon_k(G+e) < \Upsilon_k(G)$

If u & v ∉ S → S is a k dominating Set in G also which
Is not possible as $|S| < \Upsilon_k(G)$

If u & v ∈ S → by similar argument it is not possible.

Suppose u∈S and v ∉ S
Now by theorem 1 v is adjacent to exactly K vertices of S including u in Graph G+e.
So v is adjacent to exactly k-1 vertices of S in G.

6.3 Remark:

It may be noted that if u and v are two non-adjacent vertices of graph G. and if $\Upsilon_t(G+uv) < \Upsilon_t(G)$ then $\Upsilon_t(G+uv) = \Upsilon_t(G) - 1$.

6.4 *Theorem 6*:

Let u and v be two non-adjacent vertices of G and e=uv then $\Upsilon_t(G+e) < \Upsilon_t(G)$
⇔
There is a subset S of V(G) such that |S|< $\Upsilon_t(G)$ and following three conditions are satisfied.

(1) If $u \in S$ *and* $v \notin S$ then $N(v) \subset V(G) \setminus S$
(2) If $u, v \in S$ *then* $N(u) \cap S = \phi$ *or* $N(v) \cap S = \phi$
(3) If $u \notin S$ & $v \notin S$ then S is totally dominating set in $G \setminus \{u\}$ & $G \setminus \{v\}$ respectively.

Necessity:

Let S be a Υ_t Set in G+e, then $|S| < \Upsilon_t(G)$

(1) Let $u \in S$ *and* $v \notin S$ then S can not be a total dominating set in G.
Since S is total dominating set in G+e but not in G, v is not adjacent to any vertex of S in G
That is $N(v) \subset V(G) \setminus S$
(2) Suppose $u, v \in S$ then again S is not totally dominating Set in G.

Therefore either u is not adjacent to any other vertex of S or v is not adjacent to any other vertex of S
This implies that $N(u) \cap S = \phi$ *or* $N(v) \cap S = \phi$

(3) If $v \notin S$ then obviously S is a totally dominating Set in $G \setminus \{v\}$
Because u is adjacent to some vertex w of S in the Graph $G \setminus \{e\}$
Similarly if $u \notin S$ then S is a totally dominating Set in $G \setminus \{u\}$
Because v is adjacent to some vertex w of S in the Graph $G \setminus \{e\}$

Sufficiency

Let S be a subset of V(G) such that $|S| < \Upsilon t(G)$ and (1),(2),(3) are satisfied
We prove that: S is totally dominating set in G+e this will Implies that
$\Upsilon t(G+e) \leq |S| < \Upsilon t(G)$

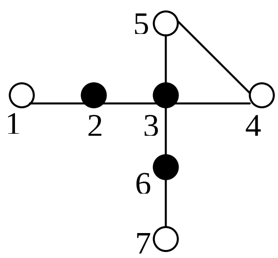
Suppose $u, v \in S$ then by condition (1) S is a totally dominating Set in G+e

Suppose $u \in S$ *and* $v \notin S$ then v is adjacent to the vertex in G+e, also Since S is totally dominating Set in graph $G \setminus \{v\}$, u is adjacent to some vertex w of S in G+e . Thus S is totally dominating Set in G+e.

Suppose $u \notin S$ & $v \notin S$
Since S is a totally dominating set in $G \setminus \{v\}$. u is adjacent to some vertex u_1 of S in $G \setminus \{v\}$
Similarly v is adjacent to some vertex v_1 of S in $G \setminus \{v\}$.
Hence S is totally dominating set in G+e.

This proves the theorem.

6.5 **Example: 3** Here we give an example of a graph which represents total dominating set and effect of edge addition between two non-adjacent vertices.



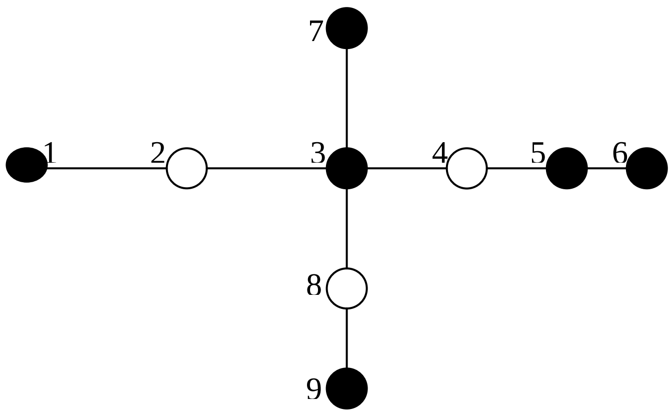
[Graph G]

Here S={2,3,6} is a Υt Set. |S|=3

In this Graph $\Upsilon t(G+e) < \Upsilon t(G)$

For e = 13, 16, 27, 37

6.6 **Example: 4** Here we give an example of a graph which represents K dominating set and effect of edge addition between two non-adjacent vertices.



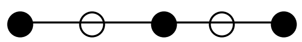
In above Graph G, Υ_2 set S={1,3,5,6,7,9} |S|=6

Here $\Upsilon_2(G+e) < \Upsilon_2(G)$

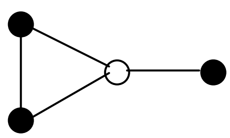
For e={17,16,36,47,57,67,69}

6.7 **Example: 5** Here we give an example of a graph in which if we add an edge between any two non-adjacent vertices then $\Upsilon_k(G+e) = \Upsilon_k(G)$ and $\Upsilon_k(G+e) < \Upsilon_k(G)$

$\Upsilon_k(G+e) = \Upsilon_k(G)$



$\Upsilon_k(G+e) < \Upsilon_k(G)$



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