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Vertex Covering Number of a Graph

[Dedicated to Late Prof. A. R. Rao]

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Abstract

In this paper we prove that the vertex covering number of a graph G does not increase when a vertex is removal from the graph. We also consider the vertex independent number of a graph. We further prove that when a vertex is removed and the vertex covering number decreases then vertex independent number remains same. On the other hand if the vertex covering number does not change then the vertex independent number decreases. In particular v Є V^-_{cr} iff v belongs to some minimum vertex covering set. Also v Є V^0_{cr} iff v does not belongs to any minimum vertex covering set of the graph G.
It may be noted that for a graph G every vertex belongs to V^0_{cr} iff the graph has no edges. Also for any cycle C_n every vertex belongs to V^-_{cr}.
Further we prove that if G a graph such that every vertex of the graph G belongs to V^-_{cr} then every vertex of the suspension of the graph G belongs to V^-_{cr}.

If a graph G is vertex transitive and has edges then every vertex belongs to V^-_{cr}.

We further prove that if G is a graph without isolated vertices and if S_1 and S_2 are disjoint vertex covering set of graph G then S_1 and S_2 are minimal vertex covering set of the graph G and G is bipartite graph.

AMS (Subject Classification): 05C70, 05C69.

Introduction

The effect of removing a vertex from a graph G on dominating sets, totally dominating sets, k-dominating sets and other such sets have been studied by several researcher [1] , [2] , [3] , [4] .

In this paper we consider so called vertex covering sets of a graph G and study them in this way we will prove that vertex covering number does not increase when a vertex is removed (unlike in the case of domination , total domination and others).

We introduce the symbols V^{-}_{cr} and V^{0}_{cr} and characterize the vertices associated with these symbols. We prove that the null graph is the only graph such that vertex covering number does not change whenever any vertex is removed. We also prove that if G is vertex transitive graph and if it has at least one edge then the vertex covering number decrease whenever any vertex is removed.

Priliminaries

We will consider only simple graphs. A null graph is a graph with at least one vertex and no edges. V(G) will denote the set of vertices and E(G) will denote the set of edges of G. If v is any vertex of the graph G then G-{v} will denote the graph obtained by removing the vertex v and all edges incident to v.

Definition 1 (Vertex Covering Set) :

Let G be a graph. A subset S of V(G) is said to be a vertex covering set of the graph G if every edge has at least one end point in S.

Definition 2(Minimal Vertex Covering Set) :

If S is a vertex covering set such that no proper subset of S is a vertex covering set then S is called minimal vertex covering set.

Definition 3 (Minimum Vertex Covering Set) :

A vertex covering set with minimum cardinality is called minimum vertex covering set. It is also called γ_{cr} set.

Definition 4 (Vertex Covering Number) :

The vertex covering number of the graph G is the minimum cardinality of any minimum vertex covering set of the graph G. It is denoted by $\alpha_0(G)$ or simply α_0.

Definition 5 (Vetex Independent Set) :

A subset S of V(G) is said to be vertex independent set if any two distinct vertices of S are non adjacent.

Definition 6 (Maximum Vetex Independent Set) :

A Vetex independent set with maximum cardinality is called maximum Vetex independent set.

Now onwards a vertex independent set will be called an independent set.

Definition 7 (Vertex Independent Number)

The cardinality of a maximum independent set is called independent number of the graph G and it is denoted by $\beta_0(G)$ or simply β_0.

Definition 8 (Vertex Transitive Graph) :

Let G be a graph then G is said to be vertex transitive if for every u,v, $\in$ V(G) there is an automorphism f(G) such that f(u) = v.

<u>Main Results :</u>

Lemma 1: If v $\in$ V(G) then

(1) $\alpha_0(G-v) \leq \alpha_0(G)$.

(2) If $\alpha_0(G-v) < \alpha_0(G)$ then $\alpha_0(G-v) = \alpha_0(G) - 1$.

Proof : (1) Case- a : v $\in$ S.

Let S be a γ_{cr} set in G and v $\in$ S. Consider the set S –{v}of G –{v}. If e = xy is an edge of G –{v} then at least one end vertex x or y lies in S. Since e is an edge of G –{v}, x $\neq$ v and y $\neq$ v. Thus, the end vertex of e which lies in S actually a vertex of S –{v}.
Thus, S- {v} is a vertex covering set of G –{v}.

Case –b : v does not belongs to S.

Here also by similar argument S is γ_{cr} set of G –{v}. Thus, $\alpha_0(G-v) \leq \alpha_0(G)$.

(3) Suppose $\alpha_0(G-v) < \alpha_0(G) - 1$.

Let S be a minimum vertex covering set of G –{v}.

Case -1 : Suppose v is not adjacent to any vertex of S.

Let S_1 = S U {v}, then S_1 is a minimal vertex covering set of the graph G. So,

$$\alpha_0(G) \leq \alpha_0(G - v) + 1 < \alpha_0(G) - 1 + 1$$

$$\text{So,} \alpha_0(G) < \alpha_0(G).$$

This is a contradiction.

Case -2 : Suppose v is adjacent to some vertex of S.

Let S_1 = S U {v}, then S_1 is a vertex covering set of the graph G. So,

$$\alpha_0(G) \leq |S_1| = |S| + 1 = \alpha_0(G - v) + 1 < \alpha_0(G) - 1 + 1$$

$$\text{So,} \alpha_0(G) < \alpha_0(G).$$

This is a contradiction.

So, by above case we have contradiction. Thus, $\alpha_0(G - v) = \alpha_0(G) - 1$. ■

Theorem 1 : Let G be a graph and v Є V(G) then v Є V^{-}_{cr} iff there is a γ_{cr} set S_1 such that v Є S_1.

Proof : Suppose that v Є V^{-}_{cr}. Let S be a minimum vertex covering set of G-{v} and let S_1 = S U {v}. Then since $\alpha_0(G - v) = \alpha_0(G) - 1$. So, S_1 is a minimum vertex covering set of the graph G. and v Є S_1.

Now we prove converse.

Let S_1 be a minimum vertex covering set of graph G containing the vertex v. Let S = S_1 – {v} then $|S| < |S_1|$. We now prove that S is a vertex covering set of the graph G – {v}. Let e = xy be an edge of the G-{v} then x ≠ v and y ≠ v. Since S_1 is a vertex covering set of the graph G so, x Є S or y Є S_1. In fact by above observation x Є S or y Є S. Thus, S is a vertex covering set of G – {v}.

$$\text{So, } \alpha_0(G - v) \leq |S| < |S_1| = \alpha_0(G)$$

$$\text{So,} \alpha_0(G - v) < \alpha_0(G)$$

So, v Є V^{-}_{cr}. ■

Corollary 1 : Let G be a graph and v Є V(G) then v Є V^{0}_{cr} iff v does not belong to any minimum vertex covering set of the graph G. ■

Suppose $S_1, S_2, \ldots S_k$ are all γ_{cr} set of the graph G

Corollary 2 : Let G e a graph and v be a vertex of the graph G. v Є V^{0}_{cr} then N(v) is subset of $S_1 \cap S_2 \cap \ldots \cap S_k$.

Proof : If N(v) = Φ then the result is obvious.

If w Є N(v) then w adjacent to v and since v does not belongs to S_i for any i (i = 1,2, 3, …..k) w Є S_i for every i . Hence w Є $S_1 \cap S_2 \cap \ldots \cap S_k$. ■

Corollary 3 : Let G be a graph and v Є V^{0}_{cr} such that v is not an isolated vertex in G then $S_1 \cap S_2 \cap \ldots \cap S_k$ is non empty. ■

Corollary 4 : The set V^{0}_{cr} is an independent set.

Proof : If u and v belongs to V^0_{cr} and if u and v adjacent then either u or v belongs to some minimum vertex covering set of the graph. In other words u Є V^-_{cr} or v Є V^-_{cr} (by theorem -1). This is a contradiction. Hence u and v are non adjacent. ■

δ(G) denote minimum degree of the graph G.

Corollary 5 : Let G be a graph then $| V^-_{cr} | \geq \delta(G)$.

Proof : If $V^0_{cr} = \Phi$ then $V^-_{cr} = V(G)$. Hence the result is true.

Suppose $V^0_{cr} \neq \Phi$. Let v Є V^0_{cr}. If v is an isolated vertex then also the result is true. Suppose v is not an isolated vertex then N(v) is a subset of $S_1 \cap S_2 \cap \ldots\ldots \cap S_k$. which is a subset of $S_1 U\ S_2 U \ldots\ldots U\ S_k$. Hence

$$\delta(G) \leq | N(v) | \leq | S_1 U\ S_2 U \ldots\ldots U\ S_k | = | V^-_{cr} |$$

Thus, $\delta(G) \leq | V^-_{cr} |$. ■

Corollary 6 : If G is any graph then $| V^0_{cr} | \leq | V(G) | - \delta(G)$. ■

Corollary 7 : For any graph G

(1) $| V^-_{cr} | \geq \alpha_0(G)$.

(2) $| V^0_{cr} | \leq \beta_0(G)$.

Proof : Note that the union of all minimum vertex covering sets = V^-_{cr} and $\alpha_0(G)$ is the cardinality of the minimum vertex covering set . It follows that $| V^-_{cr} | \geq \alpha_0(G)$ and similarly $| V^0_{cr} | \leq \beta_0(G)$. ■

Note that a graph having at least one edge has at least one non empty vertex covering set.

● We make following convention.

The graph with no edges has only one vertex covering set namely the empty set. So, vertex covering number of such a graph is zero.

Theorem 2: Let G be a graph then v Є V^0_{cr} for every vertex v Є V(G) iff the graph is a null graph.

Proof : If G is a null graph then its vertex covering number is zero and it can not decrease further when any vertex is removed. Hence every vertex belongs to V^0_{cr}.

Conversely suppose there is atleast one edge in the graph G.

Then it has a non esmpty vertex covering set. Hence any vertex x of this set belongs to V^-_{cr} by above theorem (1). This contradicts the assumption. Thus, the graph has no edges. ■

Remark :

(1) Note that the compliment of a minimum vertex covering set is a maximum independent set. Hence $\alpha_0(G) + \beta_0(G) = n$.

(2) Let v be a vertex of the graph G.

Now $\alpha_0(G\text{-}v) + \beta_0(G\text{-}v) = n -1$. If v Є V^-_{cr}, then $\alpha_0(G\text{-}v) = \alpha_0(G) - 1$,then by above equation ,

$$\alpha_0(G) -1 + \beta_0(G\text{-}v) = n\text{-}1.$$

$$\text{So, } \alpha_0 (G) + \beta_0(G-v) = n.$$
$$\text{So, } \beta_0(G-v) = n - \alpha_0 (G) .$$
$$\text{So, } \beta_0(G-v) = \beta_0(G) \text{ (So, } \alpha_0(G) + \beta_0(G) = n.)$$

Thus, we conclude that if the vertex covering number decrease (when a vertex is removed.) then the vertex independent number of the graph G does not change (when a vertex is removed.).

Similarly if the vertex covering number does not change when a vertex is removed then the vertex independent number of G is decrease (when that vertex is removed).

Example :

(1) Consider the complete graph K_n for $n \geq 2$. Its vertex covering number is n-1. For any v of K_n , $K_n - \{v\} = K_n$ -1, and its vertex covering number is n -2. Thus, every vertex of K_n belongs to V^-_{cr}.

(2) Consider the cycle C_n , $n \geq 3$ then every vertex of C_n belongs to V^-_{cr} . Similarly every vertex of the Hyper Qube Graph – Q_3 belongs to V^-_{cr}.

Theorem 3: If G is vertex transitive graph with at least one edge then every vertex v Є V^-_{cr}.

Proof : Let S be a non empty minimum vertex covering set in G and v be a vertex of the graph G. If v Є S then v Є V^-_{cr} by theorem (1)

If v does not belongs to S then let u Є S. Let f be an automorphism of the graph G such that f(u) = v (because G is vertex transitive graph). Now consider the set f(S) which is minimum vertex covering set of the graph G and it contains f(u) = v that is v Є f(S). Thus, f(S) is a minimum vertex covering set of G such that v Є f(S). So, again by theorem (1) v Є V^-_{cr}.

Thus, every vertex of the graph G belongs to V^-_{cr}.■

Theorem 4 : If G is a graph without isolated vertices and if S_1 and S_2 are disjoint vertex covering set of graph G then,

(1) G is a bipartite graph.

(2) S_1 and S_2 are minimal vertex covering set of the graph G.

Proof : (1) Let e = uv be an edge of graph G then either u Є S_1 and v Є S_2 or u Є S_2 and v Є S_1. Thus, every edge joins a vertex of S_1 to a vertex of S_2 (No edge can join two vertices of the same set of S_1 or S_2).

Moreover if x is any vertex of graph G and if e' is an edge whose one end vertex is x then x Є S_1 or x Є S_2. Thus every vertex of the graph G belongs to either S_1 or S_2. Thus, G is a bipartite graph.

(2) Now let v be any vertex of S_1 and if e is an edge whose end vertex is v then S_1-{v} does not contain the end vertex v of the edge e. Thus, S_1-{v} is not a vertex covering set of the graph G. Hence S_1 is a minimal vertex covering set of the graph G. Similarly S_2 is a minimal vertex covering set of the graph G. ■

Corollary 8 : If G is a graph without isolated vertices and if G has an odd number of vertices then any two minimum vertex cover have non empty intersection.

Proof : Suppose S_1 and S_2 are disjoint minimum vertex covering set of the graph G. Then by theorem (4) graph G is a bipartite.
Hence $|V(G)| = | S_1 | + | S_2 |$. Since $| S_1 | = | S_2 |$, $|V(G)|$ is an even number which is not true. Thus, $S_1 \cap S_2 \neq ¢$. ■

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