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Certain Generalized Properties of Hermite Polynomials

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ABSTRACT

In this research article, authors obtained certain interesting generalized properties of well-known Hermite polynomials $H_n(x)$ these shows that characteristic of Hermite polynomials as functions. Properties of Hermite polynomials in product of power of variables and Hermite polynomials of sum of power of variables also been obtained in the present paper.

Keywords: Generating function, Generating relations, Hermite polynomials

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1. INTRODUCTION

Mc Bride (1971) defined generating function as, let $F(x,t)$ be a function that can be expanded in power of t such that

$$F(x, t) = \sum_{n=0}^{\infty} c_n f_n(x) t^n \tag{1.1}$$

where c_n is a function of n that may contain the parameter of the set $\{f_n(x)\}$, but is independent of x and t. Then $F(x,t)$ is called generating function of the set $\{f_n(x)\}$.

Generating function of Hermite polynomials (Rainville(1960)) defined as

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \tag{1.2}$$

valid for all finite x and t.
Hermite polynomials also defined through Rodrigues formula (Rainville (1960)) as

$$H_n(x) = (-1)^n \exp(x^2) D^n \exp(-x^2) . \tag{1.3}$$

Many interesting results of Hermite polynomials and also its relationships (Rainville (1960)) with other well-known polynomials are mentioned.

Shukla and Prajapati (2008), Shukla et al (2010) obtained few relationships of Hermite polynomials and general class of polynomials.

In the theory of the classical orthogonal polynomials, it is fairly well known that the Laguerre polynomials $L_n^{(\alpha)}(x)$, (Chen and Srivastava (2005)) defined as

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \tag{1.4}$$

and the Hermite polynomials $H_n(x)$ defined by

$$H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \frac{(2k)!}{k!} (2x)^{n-2k} \tag{1.5}$$

Equation (1.5) is an explicit representation of Hermite polynomials.
Hermite polynomials (Szego(1975)) are also related as follows

$$H_{2n+\varepsilon}(x) = (-1)^n 2^{2n+\varepsilon} n! x^{\varepsilon} L_n^{\varepsilon-(1/2)}(x^2) \tag{1.6}$$

where ε =0 or 1.
A class of polynomials (Singhal and Srivastava (1971)), defined as

$$G_n^{[\alpha]}(x, \gamma, \beta, k) = \frac{1}{n!} x^{-\alpha-kn} \exp(\beta x^{\gamma})(x^{k+1}D)^n [x^{\alpha} \exp(-\beta x^{\gamma})] \tag{1.7}$$

Chen and Srivastava (2005) gives relation between Hermite polynomials and (1.7) as,

$$H_n(x) = \frac{n!}{(-x)^n} G_n^{(0)}(x; 2, 1, -1) \tag{1.8}$$

$$= \frac{n!}{(-x)^n} G_n^{(1-n)}(x; 2, 1, 1) \tag{1.9}$$

Well-known generating relation of Hermite polynomials defined (Rainville (1960)) as,

$$\sum_{n=0}^{\infty} H_{n+k}(x) \frac{t^n}{n!} = \exp(2xt - t^2) H_k(x-t) \tag{1.10}$$

An interesting generalization of the classical Hermite polynomials is due to (Gould and Hopper (1962)), is

$$H_n^{\gamma}(x, \alpha, \beta) = (-1)^n x^{-\alpha} \exp(\beta x^{\gamma}) D^n \{x^{\alpha} \exp(-\beta x^{\gamma})\} \tag{1.11}$$

Where $D = \frac{d}{dx}$ and the parameter α, β and γ are unrestricted, in general. In fact, in terms of classical Hermite polynomials, it is easily seen that

$$H_n^2(x, 0, 1) = H_n(x) \qquad (n = 0, 1, 2, ...) \tag{1.12}$$

2. Main Results

(i) $$H_n^{\gamma}(xy, \alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k} H_k^{\gamma}(x, \alpha, \beta)\ (-1)^{n-k}\, y^{-\alpha} \exp(\beta x^{\gamma}(y^{\gamma}-1)) D^{n-k} \left(y^{\alpha} \exp(-\beta x^{\gamma}(y^{\gamma}-1)) \right) \tag{2.1}$$

Proof: From (1.11), we can write

$$H_n^{\gamma}(xy, \alpha, \beta) = (-1)^n (xy)^{-\alpha} \exp(\beta (xy)^{\gamma}) D^n (x^{\alpha} \exp(-\beta (xy)^{\gamma})) ,\backslash$$

Consider, $H_n^{\gamma}(xy,\, \alpha,\, \beta) \; = (-1)^n x^{-\alpha} y^{-\alpha} \exp(\beta x^{\gamma} y^{\gamma}) D^n (x^{\alpha} y^{\alpha} \exp(-\beta x^{\gamma} y^{\gamma}))$

this gives

$$= (-1)^n x^{-\alpha} y^{-\alpha} \exp(\beta x^{\gamma}) \exp(\beta x^{\gamma} y^{\gamma} - \beta x^{\gamma}) \cdot$$
$$D^n \{x^{\alpha} y^{\alpha} \exp(-\beta x^{\gamma}) \exp(-\beta x^{\gamma} y^{\gamma} + \beta x^{\gamma})\}$$

$$= (-1)^n x^{-\alpha} \exp(\beta x^{\gamma}) y^{-\alpha} \exp(\beta x^{\gamma}(y^{\gamma} - 1)) \cdot$$
$$D^n \{x^{\alpha} \exp(-\beta x^{\gamma}) y^{\alpha} \exp(-\beta x^{\gamma}(y^{\gamma} - 1)\}$$

above equation can be written as

$$= (-1)^n x^{-\alpha} \exp(\beta x^{\gamma}) y^{-\alpha} \exp(\beta x^{\gamma}(y^{\gamma} - 1)) \cdot$$
$$\sum_{k=0}^{n} \binom{n}{k} D^k \left(x^{\alpha} \exp(-\beta x^{\gamma})\right) D^{n-k} \left(y^{\alpha} \exp(-\beta x^{\gamma}(y^{\gamma} - 1)\right)$$

this follows

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{-\alpha} \exp(\beta x^{\gamma}) D^k \left(x^{\alpha} \exp(-\beta x^{\gamma})\right) \cdot$$
$$(-1)^{n-k} y^{-\alpha} \exp(\beta x^{\gamma}(y^{\gamma} - 1)) D^{n-k} \left(y^{\alpha} \exp(-\beta x^{\gamma}(y^{\gamma} - 1)\right)$$

this reduces to

$$= \sum_{k=0}^{n} \binom{n}{k} H_n^{\gamma}(x,\, \alpha,\, \beta)\; (-1)^{n-k} y^{-\alpha} \exp(\beta x^{\gamma}(y^{\gamma} - 1)) D^{n-k} \left(y^{\alpha} \exp(-\beta x^{\gamma}(y^{\gamma} - 1)\right)$$

(ii) $$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \sum_{n=0}^{\infty} H_{n+k_1+k_2+\ldots+k_m}(x) \frac{t^n v_1^{k_1} v_2^{k_2} \ldots v_m^{k_m}}{k_1! k_2! \ldots k_m! n!}$$

$$= \exp(2xt - t^2) \prod_{i=1}^{m} \left(\sum_{k_i=0}^{\infty} \frac{H_{k_i}(x-t) v_i^{k_i}}{(k_i)!} \right) \exp\left(-2 \sum_{\substack{i,j=1 \\ i<j}}^{m} v_i v_j \right) \qquad (2.2)$$

Proof: Consider LHS,

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \sum_{n=0}^{\infty} H_{n+k_1+k_2+\ldots+k_m}(x) \frac{t^n v_1^{k_1} v_2^{k_2} \ldots v_m^{k_m}}{k_1! k_2! \ldots k_m! n!}$$

$$= \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k_1=0}^{n} H_{n+k_2+k_3+\ldots+k_m}(x) \frac{t^{n-k_1} v_1^{k_1} v_2^{k_2} \ldots v_m^{k_m}}{k_1! k_2! \ldots k_m! (n-k_1)!}$$

$$= \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \sum_{n=0}^{\infty} H_{n+k_2+k_3+\ldots+k_m}(x) \frac{v_2^{k_2} v_3^{k_3} \ldots v_m^{k_m}}{k_2! k_3! \ldots k_m!} \sum_{k_1=0}^{n} \frac{t^{n-k_1} v_1^{k_1}}{k_1! (n-k_1)!}$$

above can be written as,

$$=\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}...\sum_{k_m=0}^{\infty}\sum_{n=0}^{\infty}H_{n+k_2+k_3+...+k_m}(x)\frac{v_2^{k_2}v_3^{k_3}...v_m^{k_m}}{k_2!k_3!...k_m!}\frac{(t+v_1)^n}{n!}$$

$$=\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}...\sum_{k_m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k_2=0}^{n}H_{n+k_3+...+k_m}(x)\frac{v_2^{k_2}v_3^{k_3}...v_m^{k_m}}{k_2!k_3!...k_m!}\frac{(t+v_1)^{n-k_2}}{(n-k_2)!}$$

this leads to,

$$=\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}...\sum_{k_m=0}^{\infty}\sum_{n=0}^{\infty}H_{n+k_3+...+k_m}(x)\frac{v_3^{k_3}...v_m^{k_m}}{k_3!...k_m!}\sum_{k_2=0}^{n}\frac{(t+v_1)^{n-k_2}v_2^{k_2}}{k_2!(n-k_2)!}$$

this gives,

$$=\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}...\sum_{k_m=0}^{\infty}\sum_{n=0}^{\infty}H_{n+k_3+...+k_m}(x)\frac{v_3^{k_3}...v_m^{k_m}}{k_3!...k_m!}\frac{(t+v_1+v_2)^n}{n!}$$

above equation immediately leads to

$$=\sum_{n=0}^{\infty}H_n(x)\frac{(t+v_1+v_2+...+v_m)^n}{n!}$$

$$=\exp(2x(t+v_1+v_2+...+v_m)-(t+v_1+v_2+...+v_m)^2)$$

this follows,

$$=\exp(2xt-t^2)\prod_{i=1}^{m}\exp(2(x-t)v_i-v_i^2)\exp\left(-2\sum_{\substack{i,j=1\\ i<j}}^{m}v_iv_j\right)$$

we arrived at

$$=\exp(2xt-t^2)\prod_{i=1}^{m}\left(\sum_{k_i=0}^{\infty}\frac{H_{k_i}(x-t)v_i^{k_i}}{(k_i)!}\right)\exp\left(-2\sum_{\substack{i,j=1\\ i<j}}^{m}v_iv_j\right)$$

In particular

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}H_{n+k}(x)\frac{t^nv^k}{k!n!}=\exp(2xt-t^2)\left(\sum_{k=0}^{\infty}\frac{H_k(x-t)v^k}{k!}\right)\exp\left(-2(0)\right)$$

$$=\exp(2xt-t^2)\left(\sum_{k=0}^{\infty}\frac{H_k(x-t)v^k}{k!}\right). \qquad (2.3)$$

By comparing coefficient of v^k in (2.3) we have equation (1.10).

(iii) We have

$$H_n^{(\alpha+\beta)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(x)H_{n-k}^{(\beta)}(y),$$

where α considered to be a variance.

Hence,

$$H_n^{(\alpha+\beta)}(x^p+y^q)=\sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(x^p)H_{n-k}^{(\beta)}(y^q).$$

Consider,

$$H_n^{(\alpha+\beta+\gamma)}(x^p+y^q+z^r)=H_n^{(\alpha+(\beta+\gamma))}(x^p+(y^q+z^r))$$

this can be written in the form

$$=\sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(x^p)H_{n-k}^{(\beta+\gamma)}(y^q+z^r)$$

this reduces to,

$$H_n^{(\alpha+\beta+\gamma)}(x^p+y^q+z^r)=\sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(x^p)\sum_{m=0}^{n-k}\binom{n-k}{m}H_m^{(\beta)}(y^q)\,H_{n-k-m}^{(\gamma)}(z^r)$$

In general we have,

$$H_n^{(\alpha_1+\alpha_2+...+\alpha_m)}(x_1^{p_1}+x_2^{p_2}+...+x_m^{p_m})$$

$$=\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}...\sum_{k_{m-1}=0}^{n-k_1-k_2-...-k_{m-2}}\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}...\binom{n-k_1-k_2-...-k_{m-2}}{k_{m-1}}$$

$$H_{k_1}^{(\alpha_1)}(x_1^{p_1})H_{k_2}^{(\alpha_2)}(x_2^{p_2})...H_{k_{m-1}}^{(\alpha_{m-1})}(x_{m-1}^{p_{m-1}})H_{n-k_1-k_2-...-k_{m-1}}^{(\alpha_m)}(x_m^{p_m})$$

on simplification we obtained,

$$=\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}...\sum_{k_{m-1}=0}^{n-k_1-k_2-...-k_{m-2}}\frac{n!}{k_1!k_2!...k_{m-1}!(n-k_1-k_2-...-k_{m-1})!}$$

$$H_{k_1}^{(\alpha_1)}(x_1^{p_1})H_{k_2}^{(\alpha_2)}(x_2^{p_2})...H_{k_{m-1}}^{(\alpha_{m-1})}(x_{m-1}^{p_{m-1}})H_{n-k_1-k_2-...-k_{m-1}}^{(\alpha_m)}(x_m^{p_m})$$

Finally we arrived at,

$$H_n^{(\alpha_1+\alpha_2+...+\alpha_m)}(x_1^{p_1}+x_2^{p_2}+...+x_m^{p_m})$$

$$=\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}...\sum_{k_{m-1}=0}^{n-k_1-k_2-...-k_{m-2}}\frac{n!}{(n-k_1-k_2-...-k_{m-1})!}\prod_{j=1}^{m-1}\left(\frac{H_{k_j}^{(\alpha_j)}(x_j^{p_j})}{k_j!}\right)H_{n-k_1-k_2-...-k_{m-1}}^{(\alpha_m)}(x_m^{p_m}) \tag{2.4}$$

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