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Entering into Chaos for a cubic function $x^3 + \lambda x$, $-3 \le \lambda < -1.5$

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Abstract

Dynamics of a function $f(x) = x^3 + \lambda x$, $\lambda \in R$, has a different behaviour for different values of λ. This function is chaotic for $\lambda < -3$ but non-chaotic for the values $-1.5 \le \lambda < \infty$. This transition from stability to chaos for this cubic function is discussed in this paper.

Keywords: Dynamics of a function, period three points, Sarkovskii's ordering, Bifurcation Diagram

A real quadratic family [3] $f(x) = k x (1 – x)$, $k \in R$, has firmly indicated that even the simplest looking functions may have the complicated dynamics. This indication has put every function under suspicion and the simple looking cubic family functions of the form $f(x) = x^3 + \lambda x$, $\lambda \in R$ are no exceptions. It is easy to see that the dynamics of these cubic functions have simple dynamics for $-1.5 \le \lambda < \infty$. However, the dynamics of this family becomes more and more complicated as λ decreases from -1.5. Li and Yourke [8] wrote a paper "Period three implies Chaos" indicating the significance of period three points in the dynamics of a function. We shall see in this paper that period three points exist for $f(x) = x^3 + \lambda x$ as λ moves near -3 and $f(x) = x^3 + \lambda x$ turns out to be chaotic for $\lambda = -3$.

The following is a statement of Sarkovskii's theorem :

Sarkovskii's theorem (Special case) : [3] Let f be a continuous function of the real numbers and f has a periodic point with prime period 3. Then f has a periodic point of each prime period.

<u>Period 3 points for the cubic family</u>

Existence of period 3 points for the cubic family $f(x) = x^3 + \lambda x$ will give an important clue about the chaotic behaviour of the family, but to find out such points is not as easy. In fact, the period 3 points for the cubic family are contained in the set of fixed points of the polynomial equation $f^{(3)}(x) = x$. This polynomial equation is of degree 27 as given by :
$x^{27} + (9\lambda)x^{25} + (36\lambda^2)x^{23} + (84\lambda^3 + 3\lambda)x^{21} + (126\lambda^4 + 21\lambda^2)x^{19} + (126\lambda^5 + 63\lambda^3)x^{17} + (84\lambda^6 + 105\lambda^4 + 3\lambda^2)x^{15} + (36\lambda^7 + 105\lambda^5 + 15\lambda^3)x^{13} + (9\lambda^8 + 63\lambda^6 + 30\lambda^4)x^{11} + (\lambda^9 + 21\lambda^7 + 30\lambda^5 + \lambda^3 + \lambda)x^9 + (3\lambda^8 + 15\lambda^6 + 3\lambda^4 + 3\lambda^2)x^7 + (3\lambda^7 + 3\lambda^5 + 3\lambda^3)x^5 + (\lambda^6 + \lambda^4 + \lambda^2)x^3 + \lambda^3 x = x$.

Since every fixed point of f(x) also satisfies the equation $f^{(3)}x = x$, the fixed points of f(x) are the trivial solutions of the above equation of degree 27. Hence, x, $x - P_\lambda$ and $x + P_\lambda$ are factors of the 27-degree polynomial on the left hand side of the above equation. So, the period three points of f(x) are the solutions of the polynomial equation of degree 24 given by

$$x^{24} + a_1 x^{22} + a_2 x^{20} + a_3 x^{18} + a_4 x^{16} + a_5 x^{14} + a_6 x^{12} + a_7 x^{10} + a_8 x^8 + a_9 x^6 + a_{10} x^4 + a_{11} x^2 + a_{12} \quad \ldots\ldots\ldots\ldots\ldots(A)$$

Where $a_1 = 9\lambda + p^2$, $a_2 = 36\lambda^2 + p^2(a_1)$, $a_3 = 84\lambda^3 + 3\lambda + p^2(a_2)$

$a_4 = 126\lambda^4 + 21\lambda^2 + p^2(a_3)$, $a_5 = 126\lambda^5 + 63\lambda^3 + p^2(a_4)$

$a_6 = 84\lambda^6 + 105\lambda^4 + 3\lambda^2 + p^2(a_5)$, $a_7 = 36\lambda^7 + 105\lambda^5 + 15\lambda^3 + p^2(a_6)$

$a_8 = 9\lambda^8 + 63\lambda^6 + 30\lambda^4 + p^2(a_7)$, $a_9 = \lambda^9 + 21\lambda^7 + 30\lambda^5 + \lambda^3 + \lambda + p^2(a_8)$

$a_{10} = 3\lambda^8 + 15\lambda^6 + 3\lambda^4 + 3\lambda^2 + p^2(a_9)$, $a_{11} = 3\lambda^7 + 3\lambda^5 + 3\lambda^3 + p^2(a_{10})$

and $a_{12} = \lambda^6 + \lambda^4 + \lambda^2 + p^2(a_{11})$.

Solutions to the above polynomial equation can be obtained by the Sturm method [2]. This method confirms the existence of period three points for the values of λ near -3. The existence of period three points again strengthens the indications of cubic family functions becoming chaotic as they approach the value $\lambda = -3$. With these enough hints for the chaos, the next part now theoretically proves that $f(x) = x^3 + \lambda x$ is chaotic for $\lambda = -3$ on the interval [-2, 2].

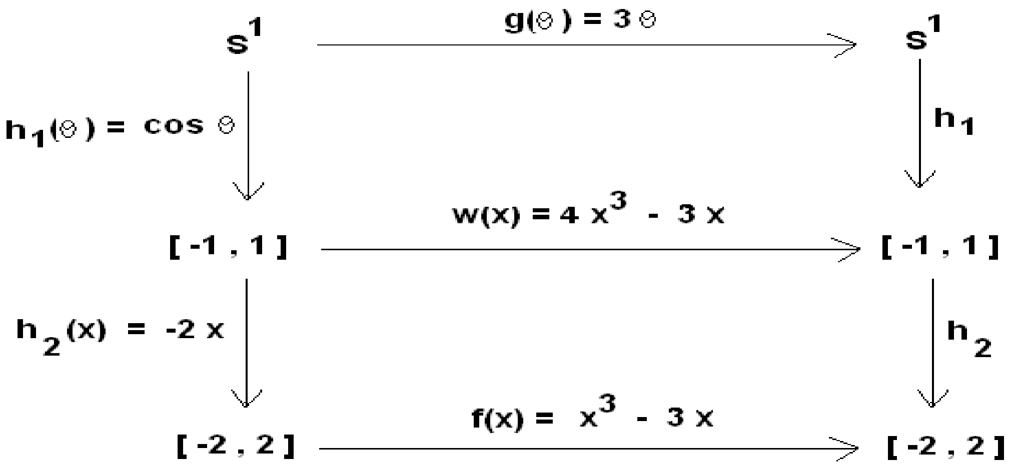
First we mention a subsidiary result (Lemma 1) which helps to prove the main result :

Lemma 1 : $f : S^1 \to S^1$ defined by $f(\theta) = 3\theta$ is chaotic, where S^1 is a unit circle.

And now we move onto the main result of the chapter showing that $f(x) = x^3 - 3x$ is chaotic on [-2, 2].

Lemma : 2 $f(x) = x^3 - 3x$ is chaotic on [-2, 2].

Part 1 : Define the functions g, h_1, h_2 and w as follows :



First observe that all these five functions are onto.

Now we have $(h_1 \circ g) = (w \circ h_1)$ ………………… (1)

And $(f \circ h_2) = (h_2 \circ w)$ …………………(2)

Further we can see that $h_1 \circ g^n = w^n \circ h_1$ for each $n \in N$. …………(3)

Similarly we have : $f^n \circ h_2 = h_2 \circ w^n$ for each $n \in N$. ………..(4)

[Note : Now onwards, composition of any two functions f and g will be expressed as fg rather than f o g.]

Now let U and V be two open intervals in [-2 , 2]. Let U' and V' be two open arcs on S^1 such that U' and V' are mapped onto U and V under $h_2 h_1$. ……………(5)

But there exists k such that g^k (U') ∩ V' ≠ ϕ. (from part 2, lemma 1)

Hence using (3) and (5), we get $f^k(U) \cap V \neq \phi$.

Thus for any two open intervals U and V in [-2 , 2], $f^k(U) \cap V \neq \phi$. This proves that f is topologically transitive in [-2 , 2].

Part 2 : To prove the sensitivity dependence of f(x) :

We want to show that there exists a $\delta > 0$ such that for any $x \in [-2, 2]$ and any neighbourhood N' of x, there is some $y \in N'$ and a positive integer n such that $|f^n(x) - f^n(y)| > \delta$.

First we prove that for any open interval U of [-2, 2], there exists some positive integer n such that $f^n(U)$ 'covers' the interval [-2 , 2]. i.e. $f^n(U) = [-2, 2]$.

Since U is an open interval in [-2 , 2], there exists an arc $(\theta_1, \theta_2) = U'$ on S^1 such that $(h_2 h_1)(U') = U$.

But $g(\theta) = 3\theta$ and hence an angular distance between any two points on S^1 is tripled under an iteration of g. So, there exists some positive integer n such that g^n covers all of S^1. i.e. there exists some positive integer n such that $g^n(U') = S^1$. …………..(6)

Now by (3) and (4), we have $h_1 \circ g^n = w^n \circ h_1$ and $f^n h_2 = h_2 w^n$.

Hence, $(f^n)(h_2 h_1) = (h_2 h_1) g^n$ ……………….(7)

By (6) $[(h_2 h_1) g^n](U') = (h_2 h_1)\{S^1\} = [-2, 2]$. ………………(8)

So, by (7) and (8), $(f^n)(h_2 h_1)(U') = f^n(U) = [-2, 2]$

Thus $f^n(U)$ 'covers' the interval [-2 , 2].

Now let $x \in [-2, 2]$ and U be a neighbourhood of x. Then there exists some $n \in N$ such that $f^n(U) = [-2, 2]$.

Since $x \in U$, $f^n(x) \in [-2, 2]$. Suppose $f^n(x) \in [-2, 0]$. Since $f^n(U) = [-2, 2]$, there exists some $y \in U$ such that $f^n(y) = 2$. Thus, $|f^n(x) - f^n(y)| > 2$.

Similarly, if $f^n(x) \in [0,2]$ then choose some $y \in U$ such that $f^n(y) = -2$. Thus, $|f^n(x) - f^n(y)| > 2$.

If $f^n(x) = 0$, choose y such that $f^n(y) = 2$ to get $|f^n(x) - f^n(y)| = 2$.

Thus, in any case, $|f^n(x) - f^n(y)| \geq 2$.

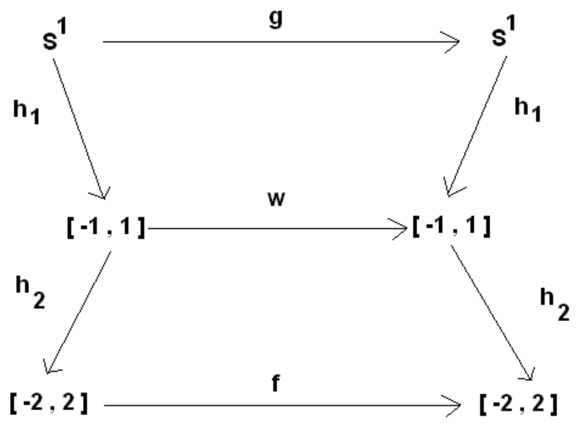
This means that there exists $\delta = 1$, say, such that $|f^n(x) - f^n(y)| > \delta$.

Part 3 : To show that periodic points are dense in [-2 , 2] :

First we show that if θ is a periodic point of period n under g, then $(h_2 h_1)(\theta)$ is also a periodic point of period n in [-2 , 2].

Let θ be a periodic point of period n under g.

So, $g^n(\theta) = \theta$.



Let $h_1(\theta) = \cos \theta = x$, then $x \in [-1, 1]$.

We first claim that x is a periodic point of period n in [-1 , 1] for function w.

We now can see that $w^n(x) = x$.

Thus, x is a periodic point of period n for a function w.

Next we show that f has a periodic point of period n :

Let $h_2(x) = y$, then $y \in [-2, 2]$.

Also $(h_2 h_1)(\theta) = h_2(h_1(\theta)) = h_2(x) = y$.

Now $f^n(y) = y$.

So, $y = (h_2 h_1)(\theta)$ is a periodic point of period n under f in [-2 , 2]. ....(9)

Finally, we show that periodic points of f are dense in [-2 , 2] :

From lemma 1 above, periodic points of a function g are dense in S^1.

Let U be any interval in [-2 , 2]. Then there exists an arc U' on S^1 such that $(h_2 h_1)(U') = U$. Since periodic points of g are dense in S^1, there exists some periodic point of period n for g in U'. Let θ be this periodic point of g with period n such that $\theta \in U'$.

From (9), $y = (h_2 h_1)(\theta)$ is a periodic point of period n under f in [-2 , 2].

Since $(h_2 h_1)(U') = U$ and $\theta \in U'$, $(h_2 h_1)(\theta) \in U$. i.e. $y \in U$.

So, y is a periodic point of period n for the function f and $y \in U$.

Thus, for any interval U in [-2 , 2], $\exists$ some point $y \in U$ which is a periodic point of f. Hence, periodic points of f are dense in [-2 , 2].

From the above three parts, it follows that $f(x) = x^3 - 3x$ is chaotic on [-2, 2].

- Thus, dynamics of $f(x) = x^3 + \lambda x$ changes to a great extent as the values of λ travel towards -3. At $\lambda = -3$, $f(x) = x^3 + \lambda x$ is chaotic on [-2, 2]. However, this in not the end of story. As the values of λ move to the left of -3, function f shows some more interesting changes in the dynamics. In fact, for $\lambda < -3$, we get for the first time that some of the points inside [-k, k] , where –k and k are the non-zero fixed points of f(x), are mapped outside of the interval [-k, k]. Hence these points approach to $-\infty$ or $+\infty$ under iterations of f. This creates an interesting situation for the dynamics of f, which we have discussed in another paper.

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