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A Study of C-Reducible Finsler Space

With Special Cases

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Abstract

The notion of C-reducible Finsler Space has been introduced by M.Matsumoto(1972).The object of the paper is to study different types of generalized C-reducible Finsler spaces and also to study the properties of hypersurfaces immersed in C-reducible Finsler space.I have also worked out for T-conditions of Finsler space & show that the tensor T_{ijkh} vanishes if & only if the tensor T_{ijkhl} vanishes for semi C-reducible Finsler spaces under certain conditions.

Keywords: Finsler Space, C-reducible, Quasi C-reducible, Semi C-reducible Finsler Space

1. Intoduction

Let α and β be two independent functions given by $\alpha(=\alpha(x,y))=(a_{ij}\ (x)\ y^i y^j)^{1/2}$, $\beta(=\beta(x,y))= b_{ij}(x)y^i y^j)^{1/2}$, $(y^i = dx^i)$, i, j= 1,......N, where a_{ij} (x), $b_{ij}(x)$ are components of symmetric tensor fields of (0,2)-type, depending on the position x alone. Then we obtain an N (≥2)-dimensional Finsler space F^N with a metric $L= L\ (\alpha(x,y),\ \beta(x,y))$, where $L(\alpha,\ \beta)$ is a positively homogeneous function of degree 1 with respect to two valuable α and β. ***Definition:***

A non-Riemannian Finsler space is called generalized C-reducible, if the Cartan torsion tensor C_{ijk} is written in the form

$$C_{ijk} = P_{ij}\, Q_k + P_{jk}Q_i + P_{ki}\, Q_j$$

It is noted that quasi-C-reducible, semi-C-reducible or C-reducible Finsler spaces are special examples of generalized C-reducible Finsler spaces.

Differentiating $L = L(\alpha,\ \beta)$ by y^i, y^j and then y^k successively, it follows from $\dot{\partial}_i\alpha = \alpha^{-1}a_{i0}, \dot{\partial}_i\beta = \beta^{-1}b_{i0}$ that the normalized supporting element l_i, the angular metric tensor h_{ij} and the torsion tensor C_{ijk} are respectively given by

(1.1) $l_i = Aa_{io} + Bb_{io}$,

(1.2) $h_{ij} = L\ \{Aa_{ij} + Bb_{ij} + A_1a_{io}a_{jo}+A_2\ (a_{io}b_{jo}+a_{jo}b_{io}) + B_2b_{io}\ b_{jo}\}$, (1.3) $$C_{ijk} = S_{(ijk)} \{pa_{ij}a_{ko} + q(a_{ij}b_{ko}+b_{ij}a_{ko}) + rb_{ij}b_{ko} + (s/3)\, a_{io}a_{jo}a_{ko} + ta_{io}a_{jo}b_{ko} + ua_{io}b_{jo}b_{ko} + (v/3)\, b_{io}b_{jo}b_{ko}\}/2$$

where we put

$$\begin{cases} A = L_\alpha \alpha^{-1},\ B = L_\beta \beta^{-1}, \\ A_1 = (L_{\alpha\alpha} - L_\alpha \alpha^{-1})\alpha^{-2},\ A_2 = L_{\alpha\beta}\, \alpha^{-1}\beta^{-1},\ B_2 = (L_{\beta\beta} - L_\beta\, \beta^{-1})\beta^{-2}, \\ A_{11} = (L_{\alpha\alpha\alpha} - 3L_{\alpha\alpha}\alpha^{-1} + 3L_\alpha \alpha^{-2})\alpha^{-3}, \\ A_{12} = (L_{\alpha\alpha\beta} - L_{\alpha\beta}\alpha^{-1})\alpha^{-2}\beta^{-1}, \\ A_{22} = (L_{\alpha\alpha\beta} - L_{\alpha\beta}\, \beta^{-1})\, \alpha^{-1}\beta^{-2} \\ B_{22} = (L_{\beta\beta\beta} - 3L_{\beta\beta}\, \beta^{-1} + 3L_\beta \beta^{-2})\, \beta^{-3} \\ \\ \mathrm{p} = A^2 + LA_1,\ q = AB + LA_2,\ r = B^2 = B^2 + LB_2, \end{cases}$$

$$s = 3AA_1 + LA_{11},\ t = BA_1 + 2AA_2 + LA_{12},$$

$$u = AB_2 + 2BA_2 + LA_{22},\ v = 3BB_2 + LB_{22}.$$

Here it is noted that the following identities hold

(1.5) $$\begin{cases} A\alpha^2 + B\beta^2 = L,\ A_1\alpha^2 + A_2\beta^2 = -A,\ A_2\alpha^2 + B_2\beta^2 = -B, \\ A_{11}\alpha^2 + A_{12}\beta^2 = -3A_1,\ A_{12}\alpha^2 + A_{22}\beta^2 = -3A_2, \\ A_{22}\alpha^2 + B_{22}\beta^2 = -3B_2,\ \mathrm{p}\alpha^2 + q\beta^2 = 0,\ q\alpha^2 + r\beta^2 = 0 \\ s\alpha^2 + t\beta^2 = -2p,\ t\alpha^2 + u\beta^2 = -2q,\ u\alpha^2 + v\beta^2 = -2r \end{cases}$$

Then we easily have

Proposition 1.1 :

The scalar q (or equivalently p,r) of the space F^N which is non-Riemannian does not vanish identically.

In virtue of above identities in (1.5) equation (1.3) is reduced to

(1.6) $C_{ijk} = P_{ij} Q_k + P_{jk} Q_i + P_{ki} Q_j,$

where we put

$P_{ij} = \alpha^{-2}\beta^{-2} \{-q(\beta^2 a_{ij} - \alpha^2 b_{ij}) + (s/3)\, \alpha^2 a_{io}\, a_{jo} + (1/6)\, (t\alpha^2 + u\beta^2)\, (a_{io}b_{jo} + a_{jo}b_{io})$

$- (v/3)\, \beta^2\, b_{i0}\, b_{jo}\}/2,$

$Q_i = \beta^2\, a_{io} - \alpha^2 b_{io}$

and $P_{io} = Q_o = 0$ hold.

2. C-reducible Finsler Spaces :

In this section, we shall deal with the C-reducible Finsler Spaces which are defined by M. Matsumoto **[2].**

Definition :

An n (≥ 3)-dimensional Finsler Space F^n is called C-reducible if the $h(hv)$-torsion tensor C_{ijk} is written in the form

(2.1) $C_{ijk} = (h_{ij}C_k + h_{jk}\, C_i + h_{ki}C_j)/(n+1).$

Theorem 2.1 :

An n (≥3) –dimensional Finsler space is C-reducible if an only if the following equation is satisfied:

(2.2) $C^2 = 3c^2/(n+1),$

where $C^2 = C_{ijk}\, C^{ijk}$ *and* $c^2 = C_iC^i.$

Proof :

Using $C_{ijk}\, h^{jk} = C_i$ and $h_{ij}C^j = C_i$

we have

$[C_{ijk} - (C_ih_{jk} + C_jh_{ki} + C_kh_{ij})/(n+1)]\, [C^{ijk} - (C^ih^{jk} + C^jh^{ki} + C^kh^{ij})/(n+1)]$

$= C^2 - 3c^2/(n+1)$

Hence the theorem is proved.

Theorem 2.2 :

An n (≥3) –dimensional Finsler space F^n is C-reducible if and only if the following equation is satisfied.

(2.3) $$S_{hijk} = [c^2 (h_{kh}h_{ij} - h_{hj}h_{ik}) + C_h C_k h_{ij} + C_iC_jh_{hk} - C_hC_jh_{ik} - C_iC_kh_{hj}]/(n+1)^2$$

Proof :

Assume that F^n is C-reducible. Then, it is known that (2.3) is satisfied **[3]**. Conversely, assume that (2.3) is satisfied. Then we have

$$C_{hkr}C^r_{ij} - C_{hjr}C^r_{ik} = \left[c^2(h_{hk}h_{ij} - h_{hj}h_{ik}) + C_hC_kh_{ij} + C_iC_jh_{hk} - C_hC_jh_{ik} - C_iC_k h_{hj}\right] / (n+1)^2$$

Contraction of this by $g^{hj}g^{ik}$ leads us to $C^2 - c^2 = -(n\text{-}2)\, c^2/(n+1)$

Hence we have $C^2 = 3c^2/(n+1)$.

By Theorem 2.1, F^n is C-reducible.

3. Quasi-C-reducibility :

Let M^n be an n-dimensional Finsler space with the fundamental function $L(x,y)$, the metric tensor g_{ij}, the angular metric tensor $h_{ij} = g_{ij} - l_il_j$ ($l_i = \partial L/\partial y^i$) and the (h) hv-torsion tensor C_{ijk}. We assume that the torsion vector $C_i = g^{jk} C_{ijk}$ has non-zero length C and the dimension n is greater than two. Then, for instance, the v-covariant derivative of a tensor K_{ij} is denoted by $K_{ij}|_k$ and the indicatrization of K_{ij} by $^/(K_{ij}) = K_{rs}\, h_i^r\, h_j^s$.

We are concerned with some special forms of C_{ijk}. First of all, it is well known that C_{ijk} in any two-dimensional Finsler space is written in the form

(3.1) $$C_{ijk} = C_iC_jC_k/ C^2$$

Definition 1 :

A Finsler space is called C2-like, if the (h) hv-torsion tensor C_{ijk} is written in the form (3.1).

Definition 2 :

A Finsler space is called semi-C-reducible, if C_{ijk} is written in the form

(3.2) $C_{ijk} = [p/(n+1)]\,(h_{ij}\, C_k + h_{jk}\, C_i + h_{ki}\, C_j) + (q/C^2)\, C_i C_j C_k.$

In the definition of semi-*C*-reducibility we do not assume that neither *p* nor q vanishes **[5]** so that semi-*C*-reducibility can include C-reducibility and *C*2-likeness. Finally, we give the definition of quasi-*C*-reducibility which was proposed by M. Matsumoto **[4] .**

Definition 4.

A Finsler space is called quasi-C-reducible, if C_{ijk} is written in the form

(3.3) $C_{ijk} = A_{ij}\, C_k + A_{jk}\, C_i + A_{ki}\, C_j,$

where A_{ij} is a symmetric and indicatric tensor.

It is immediately seen that quasi-C-reducibility is a generalization of semi-C-reducibility.

Contracting (3.4) by g^{jk}, we obtain

(3.4) $A_{ik}\, C^k = \mu C_i,$

where $\mu = (1\text{-}A)/2$ and $A = A_{jk}\, g^{jk}$. From (3.1), (2.1) and (3.3), when M^n is *C*-reducible, $A_{ij} = h_{ij}/(n+1)$. And when M^n is C2-like, $A_{ij} = C_i\, C_j\, /3C^2$. The converses of these statements are true apparently and from (3.4) we have $\mu=1/(n+1)$ and $\mu = 1/3$ respectively.

4. Hypersurfaces of a C-reducible Finsler Space

Consider a non –Riemannian hypersurface F_{n-1} of F_n (n>=4) , characterized by the equation $x^i = x^i(u^\alpha)$. The fundamental tensor of F_{n-1}

$$g_{\alpha\beta}(u,\dot{u}) = g_{ij}(x,\dot{x})B^i_\alpha B^j_\beta \qquad (4.1)$$

where $\wedge_{\alpha\gamma\beta} = g_{\gamma\delta}(\; B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$

A calculation based on the well known relation

$$C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma \qquad (4.2)$$

Above equations and the facts

$h_{\alpha\beta} = g_{\alpha\beta} - l_\alpha l_\beta, l_\alpha = B^i_\alpha l_i \; h_{\alpha\beta} = g_{\alpha\beta} - l_\alpha l_\beta, l_\alpha = B^i_\alpha l_i$ gives

$$C_{\alpha\beta\gamma} = h_{\alpha\beta} C_\gamma + h_{\beta\gamma} C_\alpha + h_{\alpha\gamma} C_\beta \qquad (4.3)$$

Where

$$C_\alpha = C_i B^i_\alpha = \frac{g^{\beta\gamma} C_{\alpha\beta\gamma}}{n}$$

This proves the following:

Theorem 4.1—A hypersurface of a C-reducible Finsler space is a C- reducible Finsler space.The difference between the intrinsic and induced connection parameters $\overset{\wedge}{\Gamma}{}^\alpha_{\beta\gamma}$ and $\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ of a hypersurface has been obtained by Rund(1965). If the space F_n is c- reducible then this difference tensor $\wedge_{\alpha\gamma\beta} = g_{\gamma\delta}(\overset{\wedge}{\Gamma}{}^\delta_{\alpha\beta} - \overset{*}{\Gamma}{}^\delta_{\alpha\beta})$ reduces to the form:

$$\wedge_{\alpha\beta\gamma} = \rho\left[h_{\beta\gamma}(\Omega_{\alpha 0} - C_\alpha \Omega_{00}) + h_{\alpha\gamma}(\Omega_{\beta 0} - C_\beta \Omega_{00}) - h_{\alpha\beta}(\Omega_{\gamma 0} + C_\gamma \Omega_{00}) \right] \qquad (4.4)$$

Where $\Omega_{\alpha 0} = 0$ $\Omega_{\alpha\beta}$ are the components of second fundamental tensor of F_{n-1}, $\rho = C_i N^i$ and N^i are the components of the unit vctor normal to the Hypersurface. This equation proves the following:

Theorem 4.2—The necessary and sufficient condition that intrinsic and induced connection parameters of a hypersurface of a C-reducible Finsler space be equal is that either $\Omega_{\alpha 0} = 0$ or the vector C_i is tangential to the surface.

5. On the Indicatrized Tensor T_{ijkhl} :

Let M^n be an *n*-dimensional Finsler space with the fundamental function $L(x,y)$ where x is a point of M^n and y is an element of support. If we denote $g_{ij}\,(x,y)$ the components of the metric tensor derived from $L(x,y)$, the angular metric tensor h_{ij} and (*h*) *hv*-torsion tensor C_{ijk} are given as

$h_{ij} = g_{ij} - l_i\, l_j$, $C_{ijk} = ½\, \partial g_{ij}/\partial y^k$ respectively ,

where $l_i = \partial L/\partial y^i$ in the normalized element of support. Then, for instance, the v-covariant derivative of the tensor K_j^i and the indicatrization of the tensor K_{ijk} are defined respectively as follows :

(5.1) $K^i_j|_k = \partial K^i_j / \partial y^k + K^r_j C^i_{rk} - K^i_r C^r_{jk},$

(5.2) $^{/}(K_{ijk}) = K_{rst} h^r_i h^s_j h^t_k,$

where

$$h^r_i = g^{rj} h_{ji} = \delta^r_i - l^r l_i, \; l^i = g^{ij} l_j$$

We define the indicatrized tensors of $C_{ijk}|_h$ and $S_{ijkh}|_l$ by the following equations and denote these tensors by T_{ijkh} and T_{ijkhl} respectively :

(5.3) $T_{ijkh} = {}^{/}(C_{ijk}|_h)$

$= C_{ijk}|_h + L^{-1} (C_{ijk} l_h + C_{hjk} l_i + C_{ihk} l_j + C_{ijh} l_k),$

(5.4) $T_{ijkhl} = {}^{/}(S_{ijkh}|_l)$

$= S_{ijkh}|_l + L^{-1}(2S_{ijkh} l_l + S_{ljkh} l_i + S_{ilkh} l_j + S_{ijlh} l_k + S_{ijkl} l_h).$

From the identities

(5.5) $S_{ijkh} = C_{ihm} C^m_{jk} - C_{ikm} C^m_{jh},$

We find that

(5.6) $S_{ijkh}|_l = C_{ihm}|_l C^m_{jk} + C_{ihm} C^m_{jk}|_l - C_{ikm}|_l C^m_{jh} - C_{ikm} C^m_{jh}|_l$

Indicatrizing the above equation using the properties of indicatrization **[1]**, we obtain

(5.7) $T_{ijkhl} = T_{ihml} C^m_{jk} + T_{jkml} C^m_{ih} - T_{ikml} C^m_{jh} - T_{jhml} C^m_{ik}.$

It follows from (5.7) that

Theorem 5.1 :

If the tensor T_{ijkh} vanishes, then the tensor T_{ijkhl} also vanishes.

Theorem 5.2 :

Let F_{ijkhl} and G_{ijkhl} be defined as follows :

(5.8) $F_{ijkhl} = 2\,(h_{ih}\,h_{jk} - h_{ik}\,h_{jh})\,C_l + (h_{lh}\,h_{jk} - h_{lk}\,h_{jh})\,C_i$

$+ (h_{ih}\,h_{lk} - h_{ik}\,h_{lh})\,C_j + (h_{ih}\,h_{jl} - h_{il}\,h_{jh})\,C_k$

$+ (h_{il}\,h_{jk} - h_{ik}\,h_{jl})\,C_h,$

$G_{ijkhl} = (h_{ih}C_jC_k + h_{jk}C_iC_h - h_{ik}\,C_jC_h - h_{jh}\,C_iC_k)\,C_l.$

Then the scalars s and t satisfying the equation

(5.9) $sF_{ijkhl} + tG_{ijkhl} = 0$

must be equal to zero, provided that M^n is of dimension $n > 3$ and C is non-zero.

Proof:

Contracting (7.9) with g^{ih} and C^l, we obtain

$$C^2\,\{2(n\text{-}1)s + C^2t\}\,h_{jk} + (n\text{-}3)\,(2s+C^2t)\,C_jC_k = 0.$$

From the above equation we have

$2(n\text{-}1)s + C^2t = 0$ and $2s + C^2t = 0$, which give $s = t = 0$

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