Mathematics Today Vol.27(June-2011)(Proc. of Maths Meet - 2011) 55-63
ISSN 0976-3228

Controllability of Second Order Nonlinear Volterra Integrodifferential Equation with nonlocal conditions

Falguni S. Acharya
Department of Applied Sciences and Humanities,
Parul Institute of Engineering and Technology,
Limda-391760, Vadodara, Gujarat, INDIA
E-mail: $falguni\_69@yahoo.co.in$

Abstract

In this paper, we prove sufficient conditions for controllability of second order nonlinear Volterra integrodifferential equation with nonlocal conditions in Banach spaces using the theory of strongly continuous cosine families and the Banach fixed point theorem.

Keywords and phrases: Controllability, Volterra integrodifferential equation, Cosine family, Banach fixed point theorem.

AMS subject Classifications: 34A37, 34G20, 34K20, 93B05, 93C10

1 Introduction

In this paper we prove controllability of second order nonlinear Volterra integrodifferential equation with nonlocal conditions in Banach spaces of the form:

$$\begin{cases} x''(t) = Ax(t) + Bu(t) + f(t, x(t), \int_0^t g(t,s,x(s))ds, x'(t)); \quad t \in J := [0,b]; \\ x(0) = x_0 + q(x, x'); \\ x'(0) = y_0 + p(x, x'); \end{cases} \tag{1.1}$$

where the state $x(.)$ takes values in a Banach space X with the norm $||.||$. A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ and the control function $u(.) \in L^2(J, U)$, with U as a Banach space. B is a bounded linear operator from U into X. Let $C = C(J, X)$ be a Banach space of all continuous functions from J into X, endowed with the norm

$$\|x\|_b = sup\{\|x(t)\| : x \in C\}, t \in J.$$

$f : J \times X \times X \times X \to X$; $g : J \times J \times X \to X$; $q, p : C^2 \to X$ are appropriates continuous functions and x_0, y_0 are given elements in X.
As indicated in [6] and reference therein, the nonlocal Cauchy problem can be applied in different fields with better effect than the classical initial condition $x(0) = x_0$. For example in [11] the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

$$g(x) = \sum_{i=0}^{p} c_i x(t_i),$$

where $c_i, i = 0, 1, ...p$ are given constants and $0 < t_0, t_1, ...t_p < b$. In this case the above equation allows the additional measurement at t_i, $i = 0, 1, ...p$. In the past several years theorems about controllability of differential, integro-differential, fractional differential systems and inclusions with nonlocal conditions have been studied by Chalishajar [7], Benchohra and Ntouyas ([3],[4]) and Hernandez, Rabello and Henriquez [13] and the references therein.
Recently, there has been increasing interest in studying the problem of controllability of nonlinear systems. Balasubramaniam et al. [2] considered a class of semilinear functional integrodifferential equations in Banach space setting and provided sufficient condition of controllability. Balachandran et al. [1] established sufficient conditions for the null controllability of nonlinear functional differential systems. Dauer and Balasubramaniam [10] established sufficient conditions of null controllability of semilinear integrodifferential systems in Banach spaces. Chalishajar [8] studied the controllability of mixed Volterra-Fredholm type integrodifferential systems in Banach spaces by applying a fixed point theorem due to Leray-Schauder alternative. Hernandez [12] proved the existence of a second order partial differential equation with nonlocal conditions by using contraction mapping principle and the cosine family theory.
The aim of this paper is to study the controllability of second order nonlinear Volterra integrodifferential equation with nonlocal conditions in Banach spaces using the theory of strongly continuous cosine families and the contraction mapping principle.

2 Preliminaries

We say that one-parameter family $\Big\{C(t) : t \in R\Big\}$ of bounded linear operators in $B(X)$ is a strongly continuous cosine family if and only if

1. $C(0) = I$, I is the identity operator on X.
2. $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $s, t \in R$.
3. the map $t \mapsto C(t)x$ is strongly continuous in t on R for each fixed $x \in X$.

The strongly continuous sine family $\{S(t) : t \in R\}$, associated to the strongly continuous cosine family $\{C(t) : t \in R\}$ is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in R.$$

Assume the following condition on A
(H1) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$, which is compact for $t > 0$; of bounded linear operators X into itself and the adjoint operator A^* is densely defined, i.e. $\overline{D(A^*)} = X^*$ (see [5]).
The infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ is the operator $A : D(A) \subset E \to X$ defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0}, \ x \in D(A),$$

where $D(A) = \{x \in X : C(.)x \in C^2(R, X)\}$, endowed with the norm

$$\|x\|_A = ||x|| + ||Ax||, x \in D(A).$$

Define $E = \{x \in X : C(.)x \in C^1(R, X)\}$. It was proved by Kisynski [14] that E endowed with the norm

$$\|x\|_E = ||x|| + \sup_{0 \le t \le 1} ||AS(t)x||, \ \ x \in E$$

is a Banach space. The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & AC(t) \end{bmatrix}$ is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\widetilde{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this, it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$ as $t \to 0$ for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is locally integrable, then $x(t) = \int_0^t C(t-s)x(s)ds$ defines an E-valued continuous function. This assertion is a consequence of the fact that

$$\int_0^t \mathcal{H}(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)ds \\ \int_0^t C(t-s)x(s)ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.
For the system (1.1) we assume that the following hypotheses are satisfied:
(H2) Let $W : L^2(J, U) \to X$ be the linear operator defined by

$$Wu = \int_0^b S(b-s)Bu(s)ds$$

The $W : L^2(J, U)/kerW \to X$ induces a bounded invertible operator $\widetilde{W}^{-1}$ and there exists positive constant M_1 and M_2 such that and $\|B\| \le M_1$ and $\|\widetilde{W}^{-1}\| \le M_2$.
(H3) $f : J \times X \times X \times X \to X$ is continuous in t on J and there exist positive constants $L_f^i, i = 1, 2, 3$ such that

$$\|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\| \le L_f^1 \|x_1 - x_2\| + L_f^2 \|y_1 - y_2\| + L_f^3 \|z_1 - z_2\|$$

(H4) $g : J \times J \times X \to X$ is continuous in t, s on J and there exist positive constant $K > 0$ such that

$$\|g(t, s, x_1) - g(t, s, x_2)\| \le K \|x_1 - x_2\|$$

(H5) $q, p : C \times C \to X$ are continuous, $q(.)$ is E valued and there exist constants $L_q^i, L_p^i; i = 1, 2$ such that

$$\|q(x_1, y_1) - q(x_2, y_2)\|_E \leq L_q^1 \|x_1 - x_2\|_b + L_q^2 \|y_1 - y_2\|_b$$

and

$$\|p(x_1, y_1) - p(x_2, y_2)\|_E \leq L_p^1 \|x_1 - x_2\|_b + L_p^2 \|y_1 - y_2\|_b$$

LEMMA 2.1 *Let X be a Banach space and $T : X \to X$ be a contraction on X. Then T has precisely one fixed point.* ∎

DEFINITION 2.2 *The system (1.1) is said to be controllable on J if for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the corresponding solution $x(\cdot)$ of (1.1) satisfies*

$$\begin{aligned} x(t) &= C(t)[x_0 + q(m,n)] + S(t)[y_0 + p(m,n)] \\ &+ \int_0^t S(t-s)(Bu)(s)ds + \int_0^t S(t-s) f\Big(s, m(s), \int_0^s g(s,\tau,m(\tau))d\tau, n(s)\Big)ds \end{aligned}$$

3 Controllability Result

We now state and prove the main controllability result.

THEOREM 3.1 *Assume that the hypotheses (H1)-(H5) are satisfied. Then the system (1.1) is controllable for x_0 and x_1 on J provided,*

$$\begin{aligned} r &= max\{r_1, r_2\}, where \\ r_1 &= M[\|x_0\| + Q] + N[\|y_0\| + P] + NM_1M_2M_3b \\ &+ N\Big\{L_f^1 rb + L_f^2 Krb^2 + L_f^2 K_1 sr^2 + L_f^3 rb + L_1 b] \end{aligned}$$

and

$$\begin{aligned} r_2 &= \rho[\|x_0\| + Q] + M[\|y_0\| + P] + MM_1M_2M_3b + M\Big\{L_f^1 rb + L_f^2 Krb^2 \\ &+L_f^2 K_1 sr^2 + L_f^3 rb + L_1 b] \\ l &= l_1 + l_2 \leq 1, where \\ l_1 &= max\{[ML_q^1 + NL_p^1 + NL_f^1 b + NL_f^2 bK + NM_1M_2b(ML_q^1 + NL_p^1 + NL_f^1 b \\ &+NL_f^2 bK)], [ML_q^2 + NL_p^2 + NM_1M_2b(ML_q^2 + NL_p^2 + NL_f^3 b) + NL_f^3 b]\} \end{aligned}$$

and

$$\begin{aligned} l_2 &= max\{[\rho L_q^1 + ML_p^1 + MM_1M_2b[ML_q^1 + NL_p^1 + NbL_f^1 + L_f^2 bK] + MbL_f^1 \\ &+Mb^2 L_f^2 K], [\rho L_q^2 + ML_p^2 + MM_1M_2b[ML_q^2 + NL_p^2 + Nb^2 L_f^3] + Mb^2 L_f^3]\}. \end{aligned}$$

Moreover,

$$L_1 = max_{t\in J}\|f(t,0,0,0)\|, K_1 = max_{t\in J}\|g(t,s,0)\|,$$
$$Q = max_{m,n\in(J,B_r)}\|q(m,n)\|, P = max_{m,n\in(J,B_r)}\|p(m,n)\|,$$

Proof: On the space $Y = C^2$, equipped with the norm

$$\|(m,n)\| = \|m\|_b + \|n\|_b.$$

Let $B_r = \{z : \|z\| \leq r\} \subset\subset X$. Let $Z = C(J, B_r) \subset C$ and define an operator $\phi : Z^2 \to Z^2$, where $\phi(m,n) = (\phi_1(m,n), \phi_2(m,n))$ by

$$\begin{aligned}\phi_1(m,n)(t) &= C(t)[x_0 + q(m,n)] + S(t)[y_0 + p(m,n)] \\ &+ \int_0^t S(t-s)(Bu)(s)ds \\ &+ \int_0^t S(t-s)f\Big(s, m(s), \int_0^s g(s,\tau,m(\tau))d\tau, n(s)\Big)ds\end{aligned}$$

and

$$\begin{aligned}\phi_2(m,n)(t) &= AS(t)[x_0 + q(m,n)] + C(t)[y_0 + p(m,n)] \\ &+ \int_0^t C(t-s)(Bu)(s)ds \\ &+ \int_0^t C(t-s)f\Big(s, m(s), \int_0^s g(s,\tau,m(\tau))d\tau, n(s)\Big)ds\end{aligned}$$

for $(m,n) \in Z^2$.
Using the hypothesis (H3) for an arbitrarily function $x(.),\ x_1 \in X$ define the control,

$$\begin{aligned}u_x(t) &= \widetilde{W}^{-1}\Big[x_1 - C(b)[x_0 + q(m,n)] - S(b)[y_0 + p(m,n)] \\ &- \int_0^b S(b-s)f\Big(s, m(s), \int_0^s g(s,\tau,m(\tau))d\tau, n(s)\Big)ds\Big]\end{aligned}$$

We shall first show that ϕ_1 is well defined.

$$\begin{aligned}\|\phi_1(m,n)(t)\| &\leq \|C(t)[x_0 + q(m,n)]\| + \|S(t)[y_0 + p(m,n)]\| + \|\int_0^t S(t-s)(Bu)(s)ds\| \\ &+\|\int_0^t S(t-s)f\Big(s, m(s), \int_0^s g(s,\tau,m(\tau))d\tau, n(s)\Big)ds\| \\ &\leq M[\|x_0\| + Q] + N[\|y_0\| + P] + NM_1M_2\int_0^t \{\|x_1\| + M[\|x_0\| + Q] \\ &+N[\|y_0\| + P] + N\int_0^b [\|f\Big(s, m(s), \int_0^s g(s,\tau,m(\tau))d\tau, n(s)\Big) \\ &-f(s,0,0,0) + f(s,0,0,0)\|]ds\}ds \\ &+N\int_0^t [\|f\Big(s, m(s), \int_0^s g(s,\tau,m(\tau))d\tau, n(s)\Big) \\ &-f(s,0,0,0) + f(s,0,0,0)\|]ds \\ &\leq M[\|x_0\| + Q] + N[\|y_0\| + P] + NM_1M_2\int_0^t \Big\{\|x_1\| + M[\|x_0\| + Q]\end{aligned}$$

$$
\begin{aligned}
& +N[\|y_0\|+P]+N\int_0^b [L_f^1\|m(s)-0\|+L_f^2\|\int_0^s g(s,\tau,m(\tau))d\tau-0\| \\
& +L_f^3\|n(s)-0\|+L_1]ds\Big\}ds+N\int_0^t [L_f^1\|m(s)-0\| \\
& +L_f^2\|\int_0^s g(s,\tau,m(\tau))d\tau-0\|+L_f^3\|n(s)-0\|+L_1]ds \\
\leq\ & M[\|x_0\|+Q]+N[\|y_0\|+P]+NM_1M_2\int_0^t\Big\{\|x_1\|+M[\|x_0\|+Q] \\
& +N[\|y_0\|+P]+N\int_0^b [L_f^1\|m(s)\| \\
& +L_f^2\int_0^s \|g(s,\tau,m(\tau))-g(s,\tau,0)\|d\tau \\
& +L_f^2\int_0^s \|g(s,\tau,0)\|d\tau+L_f^3 r+L_1]ds\Big\}ds+N\int_0^t [L_f^1\|m(s)\| \\
& +L_f^2\int_0^s \|g(s,\tau,m(\tau))-g(s,\tau,m(0))\|d\tau \\
& +L_f^2\int_0^s \|g(s,\tau,0)\|d\tau+L_f^3 r+L_1]ds \\
\leq\ & M[\|x_0\|+Q]+N[\|y_0\|+P]+NM_1M_2\int_0^t\Big\{\|x_1\|+M[\|x_0\|+Q] \\
& +N[\|y_0\|+P]+N\int_0^b\Big\{L_f^1 r+L_f^2\int_0^s Krd\tau+L_f^2K_1 s+L_f^3 r+L_1]ds\Big\}ds \\
& +N\int_0^t [L_f^1 r+L_f^2\int_0^s Krd\tau+L_f^2K_1 s+L_f^3 r+L_1]ds \\
\leq\ & M[\|x_0\|+Q]+N[\|y_0\|+P]+NM_1M_2M_3 b \\
& +N\Big\{L_f^1 rb+L_f^2 Krb^2+L_f^2K_1 sr^2+L_f^3 rb+L_1 b]\Big\}=r_1 \\
\leq\ & r
\end{aligned}
$$

for $(m,n)\in Z^2$ and $t\in J$. This shows that ϕ_1 is well defined and take values in Z. Similarly, we can prove ϕ_2 is well defined and takes values in Z.
Moreover, for $(m,n),(w,z)\in Z^2$ and $t\in J$, we get

$\|\phi_1(m,n)(t)-\phi_1(w,z)(t)\|$

$$
\begin{aligned}
\leq\ & \|C(t)\|\|q(m,n)-q(w,z)\|+\|S(t)\|\|p(m,n)-p(w,z)\| \\
& +\int_0^t NM_1M_2\Big\{M\|q(m,n)-q(w,z)\|+N\|p(m,n)-p(w,z)\| \\
& \qquad +N\int_0^b [L_f^1\|m(s)-w(s)\| \\
& \qquad +L_f^2\int_0^s \|g(s,\tau,m(\tau))-g(s,\tau,w(\tau))\|d\tau+L_f^3\|n(s)-z(s)\|]ds\Big\}ds \\
& +N\int_0^t [L_f^1\|m(s)-w(s)\|
\end{aligned}
$$

$$
\begin{aligned}
&+L_f^2\int_0^s \|g(s,\tau,m(\tau)-g(s,\tau,w(\tau)))\|d\tau + L_f^3\|n(s)-z(s)\|]ds \\
\leq\ & [ML_q^1 + NL_p^1 + NL_f^1 b + NL_f^2 bK + NM_1M_2b(ML_q^1 + NL_p^1 + NL_f^1 b + NL_f^2 bK)] \\
& \|m-w\|_b + [ML_q^2 + NL_p^2 + NM_1M_2b(ML_q^2 + NL_p^2 + NL_f^3 b) + NL_f^3 b]\|n-z\|_b \\
\leq\ & l_1\|m-w\|_b + l_1\|n-z\|_b \\
\leq\ & l_1\|m-w, n-z\| \\
\leq\ & l_1[\|(m,n)-(w,z)\|_b]
\end{aligned}
$$

Therefore,

$$\|\phi_1(m,n)(t) - \phi_1(w,z)(t)\| \leq l_1[\|(m,n)-(w,z)\|_b] \tag{3.2}$$

$\|\phi_2(m,n)(t) - \phi_2(w,z)(t)\|$

$$
\begin{aligned}
\leq\ & \|AS(t)\|\|q(m,n)-q(w,z)\| + \|C(t)\|\|p(m,n)-p(w,z)\| \\
& + \int_0^t \|C(t-s)\|M_1M_2\Big\{M\|q(m,n)-q(w,z)\| + N\|p(m,n)-p(w,z)\| \\
& \qquad + N\int_0^b [L_f^1\|m(s)-w(s)\| \\
& \qquad + L_f^2\int_0^s \|g(s,\tau,m(\tau)) - g(s,\tau,w(\tau))\|d\tau + L_f^3\|n(s)-z(s)\|]ds\Big\}ds \\
& + \int_0^t \|C(t-s)[f(s,m(s),\int_0^s g(s,\tau,m(\tau))d\tau, n(s)) \\
& - f(s,w(s),\int_0^s g(s,\tau,w(\tau))d\tau, n(s))]ds \\
\leq\ & \{\rho L_q^1 + ML_p^1 + MM_1M_2b[ML_q^1 + NL_p^1 + NbL_f^1 + L_f^2 bK] + MbL_f^1 + Mb^2L_f^2K\} \\
& \|m-w\|_b + \{\rho L_q^2 + ML_p^2 + MM_1M_2b[ML_q^2 + NL_p^2 + Nb^2L_f^3] + Mb^2L_f^3\}\|n-z\|_b \\
\leq\ & l_2\|m-w\|_b + l_2\|n-z\|_b \\
\leq\ & l_2\|m-w, n-z\| \\
\leq\ & l_2[\|(m,n)-(w,z)\|_b]
\end{aligned}
$$

Therefore,

$$\|\phi_2(m,n)(t) - \phi_2(w,z)(t)\| \leq l_2[\|(m,n)-(w,z)\|_b] \tag{3.3}$$

From (3.2), (3.3) and condition on l, it follows that

$$\|\phi(m,n) - \phi(w,z)\|_b \leq l\|(m,n)-(w,z)\|_b$$

which imply that ϕ is a contraction. Hence by the Banach fixed point theorem, the function ϕ has a unique fixed point and (1.1) is controllable on J.

4 Example

Consider the following partial differential equation:

$$\begin{cases} \frac{\partial^2 w(t,\xi)}{\partial t^2} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + u(t,\xi) + \mu\Big(t, (w(t,\xi)), \int_0^t a(t,s,w(s,\xi))ds, \frac{\partial w(t,\xi)}{\partial t}\Big), \\ t \in J, \xi \in I = [0,\pi], \\ w(t,0) = w(t,\pi) = 0, t \in J, \\ w(0,\xi) = x_0(\xi) + \int_0^\pi Q_1(w(s,.))(\xi)ds, \xi \in I, \\ \frac{\partial w(t,\xi)}{\partial t} \mid_{t=0} = y_0(\xi) + \int_0^\pi P_1\Big(\frac{\partial w(s,.)}{\partial s}\Big)(\xi)ds, \xi \in I, \end{cases} \tag{4.1}$$

where $\mu : J \times \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$, $a : J \times J \times \mathcal{R} \to \mathcal{R}$ are continuous. Let us take $X = L^2([0,\pi])$. $x_0, y_0 \in X$ and $P_1 : X \to X$, $Q_1 : X \to E$ are Lipschitz continuous. We define the operator $A : D(A) \subset X \to X$ by $Aw = w_{\xi\xi}$, where $D(A) = \{w(.) \in X : w(0) = w(\pi) = 0\}$, where A is the generator of strongly continuous cosine function $\{C(t) : t \in \mathcal{R}\}$ on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathcal{N}$, with corresponding normalized characteristics vectors $w_n(\xi) := \sqrt{\frac{2}{\pi}} sin(n\xi), n = 1, 2, 3, ...$, and the following conditions hold:
(i) $\{w_n : n \in \mathcal{N}\}$ is an orthonormal basis of X.
(ii) If $w \in D(A)$ then $Aw = -\sum_{n=1}^{\infty} n^2 < w, w_n > w_n$.
(iii) For $w \in X, C(t)w = \sum_{n=1}^{\infty} cos(nt) < w, w_n > w_n$. And $S(t)w = \sum_{n=1}^{\infty} \frac{sin(nt)}{n} < w, w_n > w_n$, that $S(t)$ is compact for every $t > 0$ and that $\|C(t)\| \leq 1$ and $\|S(t)\| \leq 1$ for every $t \in J$.
(iv) If H denotes the group of translations on X defined by $H(t)x(\xi) = x\ (\xi + t)$, where x is the extension of x with period 2π, then $C(t) = \frac{1}{2}(H(t) + H(-t))$. Hence it follows that $A = G^2$, where G is the infinitesimal generator of the group H and that $E = \{x \in L^1(0,\pi) : x(0) = x(\pi) = 0\}$.
The control operator $B : L^2(J, X) \to X$ is defined by $(Bu)(t)(\xi) = u(t,\xi); \xi \in (0,\pi)$ which satisfied condition (). Here B is an identity operator and the control function $u(.)$ is given in $L^2([0,\pi], X)$.
Define the functions $f : J \times X \times X \times X \to X$, $a : J \times J \times X \to X$, and $q, p : C(J, X)^2 \to X$ as follows:

$$f(t,x,y,z)(\xi) = \mu(t, x(\xi), y(\xi), z(\xi)),$$

$$g(t,s,x)(\xi) = a(t,s,x(\xi)),$$

$$q(m,n) = \int_0^\pi Q_1(m(s))(\xi)ds,$$

and

$$p(m,n) = \int_0^\pi P_1(n(s))(\xi)ds, m, n \in C(I : X)$$

The equation (4.1) can be formulated as an abstract nonlinear second order integrodifferential equations (1.1) in Banach spaces X by using the previous assumptions and choice of functions and generator A.By controllability theorem we can get the solution of the nonlinear partial integrodifferential equations (4.1).

References

[1] Balachandran, K., Balasubramaniam, P. and Dauer,J. P.; *Local null controllability of functional integrodifferential equaitions in Banach space;* J. Optim. Theory Appl., Vol. 88, (1996), 61-75.

[2] Balasubramaniam, P., Dauer, J. P. and Logananthan, C.; *Local controllability of functional integrodifferential equaitions in Banach space;* J. Optim. Theory Appl., Vol. 114,(2002), 273-286.

[3] Benchohra, M. and Ntouyas, S. K.; *Existence and controllability results for nonlinear differential inclusions with nonlocal conditions in Banach spaces;* J.Appl. Anal., Vol. 1,8 (2002), 45-52.

[4] Benchohra, M. and Ntouyas, S. K.; *On Second order Impulsive functional differential equations in Banach spaces;* Journal of Applied Mathematics and Stochastic Analysis, 15 (2002), 45-52.

[5] Bochenek, J., *An abstract nonlinear second-order differential equation,* Ann. Polo. Math., 54(1991), pp. 155-166.

[6] Byszewski, L.; *Theorems about the existence and uniqueness of a semilinear evolution nonlocal Cauchy problem;* J. Math. Anal. Appl., Vol. 162, (1991), 496-505.

[7] Chalishajar, D. N.; *Controllability of second order semilinear neutral functional differential inclusions with non-local conditions in Banach spaces;* Nonlinear Functional Analysis and Applications, Vol. 14(1), (2009), 1-15.

[8] Chalishajar, D. N.; *Controllability of mixed Volterra-Fredholm type integrodifferential systems in Banach spaces;* J. Fraklin Inst., Vol. 344, (2007), 12-21.

[9] Chalishajar, D. N., George, R. K. and Nandakumaran, A. K., and Acharya, F. S., *Trajectory Controllability of nonlinear integro-differential systems,* Journal of the Franklin Institute, 347(2010), pp.1065-1075.

[10] Dauer, J., P. and Balasubramaniam, P.; *Null controllabiliyt of semilinear integrodifferential systems in Banach space;*Appl. Math. Lett.; Vol. 10, (1997), 117-123.

[11] Deimling, K.; *Multivalued Differential Equations;* De Gruyter, Berlin, (1992).

[12] Hernandez, E.; *Existence of solutions to a second order partial differentail equation with nonlocal condition;* Electronic J. of Diff. Equations; Vol. 2003(51), (2003), 1-10.

[13] Eduardo Hernandez, Marco Rabello and Hernan R. Henriquez; *Existence of solutions for impulsive partial neutral functional differential equations;* Journal of Mathematical Analysis and Applications, Vol. 331, (2007), 1135-1158.

[14] Kisynski, J., *On second order Cauchy problem in a Banach Space,* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 18(1970), pp. 371-374.

[15] Pachpatte, B., G., *Inequalities for Differential and Integral equations,* Academic Press, New York, (1998).