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Big Total Domination Number and Well Totally Dominated Graphs

(Dedicated to late Prof. P. C. Vaidya)

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Abstract

In this paper we consider the big total domination number of a graph and prove that it does not increase when a vertex is removed. Further we prove the necessary and sufficient condition under which the removal of a vertex does not change the big total domination number. Also we consider well totally dominated graphs, approximately well totally dominated graphs and prove some results.

AMS Subject Classification:05C69

Key Words

Totally dominating sets, Total domination number, Big total domination number, Well totally Dominated graph, Minimal totally dominating set, Minimum totally dominating set, Approximately well totally dominated graph.

Introduction

If G is a graph and if G has a totally dominating set then its total domination number is defined to be the cardinality of the smallest totally dominating set. This number is denoted as γ_t (G). When a vertex is removed from the graph G this number may increase, decrease or remains same. D.K.Thakkar and Vala in[1] have studied this aspects and characterized those vertices whose removal increases, decreases or does not change the total domination number.

Like domination number there is associated with any graph G(without isolated vertices) a number called big total domination number. It is the cardinality of a minimal totally dominating set with biggest size. In particular we characterise those vertices whose removal increases, decreases or does not change the big total domination number.

We also consider so called well totally dominated graphs. For these graphs the total domination number is same as the big total domination number. We consider the effect of removing a vertex for these graphs. Further we defined so called approximately well totally dominating graph and proved some related results.

Preliminaries and notations

For a graph G, V(G) will denote the vertex set of the graph G. G-v will denote the sub graph obtained by removing the vertex v from the graph G.

We assume that all graphs are simple and have no isolated vertices.

DEFINITION 1 [4]

Let G be a graph and S be a subset of V(G). The set S is said to be a totally dominating set if for every vertex v of G , v is adjacent to some vertex of S.

DEFINITION 2 [4]

A totally dominating set with minimum cardinality is called a minimum totally dominating set and is called a γ_t set of the graph.

DEFINITION 3 [4]

The cardinality of a minimum totally dominating set is called the total domination number of the graph G and is denoted as $\gamma_t(G)$.

DEFINITION 4 [4]

A totally dominating set S of G is said to be a minimal totally dominating set if for every vertex v of S , S – v is not a totally dominating set .

DEFINITION 5

Minimal totally dominating set with maximum cardinality is called a Γ_t set of a graph G and the cardinality of such a set is called the big total domination number of G and it is denoted as $\Gamma_t(G)$.

DEFINITION 6 [1]

Let S be a subset of V(G) and $v \in S$ then the total private neighbourhood of v with respect to the set S is defined as

$$P_{rt}(v, S) = \{ w \in V(G) / N(w) \cap S = \{ v \} \}.$$

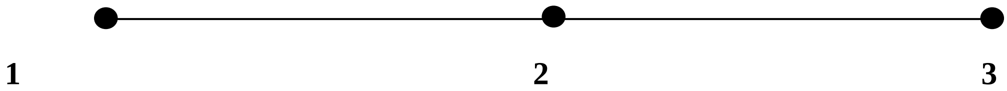

Fig. 1 : Path graph with three vertices.

In the above figure the set $\{ 2 , 3 \}$ is a minimal totally dominating set of the graph G = the path graph with three vertices. Also if $S = \{ 2 , 3 \}$ and $v = 2$ then $P_{rt}(2 , S) = \{ 1 , 3 \}$ and $P_{rt}(3 , S) = \{ 2 \}$.

$\{1,3\}$ is dominating set but not totally dominating set .

Notations

V_t^i $\{ v \in V(G) / G - v$ has an isolated vertex $\}$.

V_t^+ $\{ v \in V(G) / \gamma_t (G - v) > \gamma_t (G) \}$.

V_t^- $\{ v \in V(G) / \gamma_t (G - v) < \gamma_t (G) \}$.

V_t^0 $\{ v \in V(G) / \gamma_t (G - v) = \gamma_t (G) \}$.

W_t^+ $\{ v \in V(G) / v \notin V_t^i$ and $\Gamma_t(G - v) > \Gamma_t(G) \}$.

W_t^- $\{ v \in V(G) / v \notin V_t^i$ and $\Gamma_t(G - v) < \Gamma_t(G) \}$.

W_t^0 $\{ v \in V(G) / v \notin V_t^i$ and $\Gamma_t(G - v) = \Gamma_t(G) \}$.

The proof of the following theorem can be found in [1]

THEOREM 1

A totally dominating set S of the graph G is a minimal totally dominating set if and only if for every vertex $v \in S$, $P_{rt}(v , S)$ is a non empty set.

Main results

Now we prove that when a vertex is removed the big total domination number does not increase. Further we also give a necessary and sufficient condition under which this number does not change.

THEOREM 2

Suppose G is a graph, v ∈ V(G) such that

v $\notin V_t^i$ then Γ_t(G – v) ≤ Γ_t(G).

PROOF :

Suppose S is a Γ_t set of G – v there are three possibilities for the vertex v:
(i)v is not adjacent to any vertex of S. Let w be a vertex adjacent to thevertex v.Since , S is a totally dominating set in G – v, S_1 = S U { w } is a totallydominating set in G. In fact S_1 is a minimal totally dominating set in G.
Therefore Γ_t(G) ≥ Cardinality of S_1 = $|S_1|$ > Cardinality of S = Γ_t(G – v).

Thus Γ_t(G – v) ≤ Γ_t(G).

(ii)v is adjacent to exactly one vertex w of S.

Thus S is a minimal totally dominating set in G.

Therefore Γ_t(G) ≥ Cardinality of S = Γ_t(G – v).

(iii) v is adjacent to at least two vertices of S. Then S_1 = S U { v } is a minimal totally dominating set of G.

Therefore Γ_t(G) ≥ $|S_1|$ > $|S|$ = Γ_t(G – v).

Therefore Γ_t(G) ≥ Γ_t(G – v).

Thus in all cases Γ_t(G – v) ≤ Γ_t(G).■

THEOREM 3

Let G be a graph and v be a vertex ofV(G) such that v $\notin V_t^i$.

Then $V \in W_t^0$ if and only if either there is aΓ_t set S of G such thatvdoesnot belongs to S andv isadjacent to at least two verticesof S, or there is a$\Gamma_t$set S_1 of G such that v does not belongs to $S_1$and there is a vertex w inS1 such that the total private neighbourhood of w with respect to S1 contains at least two vertices including v.

PROOF :Suppose v ∈ W_t^0 .

Let S be a Γt set of G – v . If v is not adjacent to any vertex of S then let w be any vertex adjacent to v then , T = S U { w } is a minimal totally dominating set of graph G and |T|>|S| .
Therefore Γt (G) ≥ |T|>|S| = Γt (G - v) .

That isΓt (G - v) <Γt (G) . This means that v ∈ W_t^-
which contradicts with our assumption. Therefore v must be adjacent to some vertex of S.

Suppose there is a vertex w ∈ S such that v is adjacent to only w in S , Therefore v ∈ Prt(w , S) also S is a minimal totally dominating set in G – v . Therefore total private neighbourhood of w with respect to S in G – v contains a vertex v'.Thus Prt(w , S) contains at least two vertices and one of them is v.

In the other case , that is v is adjacent to at least two vertices of S then S is a minimal totally dominating set of G not containing v and v is adjacent to at least two vertices of S .

CONVERSE

Suppose S is a Γ_t set of G not containing v such that v is adjacent to at least two vertices of S , then for every vertex w in S. P_{rt}(w , S) can not contain v . Therefore S is a minimal totally dominating set in G – v . Therefore Γ_t (G - v) ≥ |S| = Γ_t (G) .

Since Γ_t (G - v) >Γ_t (G) is not possible , we have Γ_t (G - v) = Γ_t (G) . Hence v∈ W_t^0 .

Suppose S is a Γ_t-set such that v ∉S and there is a vertex w ∈ S such that P_{rt}(w , S) contains at least two vertices and one of them is v , therefore P_{rt}(w , S) contains a vertex of G – v . Also for other vertices w ' in S . wcan not be a member of P_{rt}(w' , S) , contains a vertex of G – v . Thus , S is a minimal totally dominating set in G – v.

By similar argument of S in above case , we have

Γ_t (G - v) = Γ_t (G) . ■

Now we give the necessary and sufficient condition under which the big total domination number decreases.

THEOREM 4

Let G be a graph and v be a vertex of G such that v ∉ V_t^i then v ∈ W_t^- if and only if , whenever S is aΓtset of G not containing v then there is a vertex w in S such that

Prt(w , S) = { v }.

PROOF :

Suppose v ∈ W_t^- . Let Sbe a Γt set of G such that v does not belongs to S.

Now v is adjacent to some vertex of S if v is adjacent to at least two vertices of S then by previous theorem v ∈ W_t^0 . Which contradictsour assumption with v ∈ W_t^- , therefore there is a vertex w in S such that v is adjacent to w and v is not adjacent to any other vertexof S,this implies that if there is a another vertex v' in G such that v' ∈ Prt(w , S) then again by previous theorem v ∈ W_t^0 . Whichis a contradiction.

Hence Prt(w , S) = { v } .

CONVERSE :

Suppose v ∈ W_t^0 then there is a Γt set S of G not containing v such that one of the following two conditions hold

(a) There is a vertex w in S such that Prt(w , S) Contains at least two vertices including v.

(b)Now v is adjacent to at least two vertices of S

Now , suppose condition (a) holds.

There is a vertex w' in s such that Prt(w' , S) = {v}.

If w = w' then our condition is violated . Suppose w ≠ w' , then v is adjacent to two vertices of S, it implies that v ∉ Prt(w' , S).

If v is adjacent to at least two vertices of S then v ∉ Prt(w' , S) for any w' in S . This again violate with our condition .

Thus v ∈ W_t^0 gives rise to a contradiction in either case , thus v ∈ W_t^- .Hencethe theorem is proved.■

DEFINITION 7

Let G be a graph then G is said to be well totally dominated graph if all minimal totally dominating sets of G have the same cardinality . Equivalently , $\gamma_t(G) = \Gamma_t(G)$.

Now we prove the following theorem about well totally dominated graphs.

THEOREM 5

Suppose G is a well totally dominated graph and v ∈ V(G) such that v ∉ V_t^i then the following statements are true.

(i) v ∉ V_t^+ (That is V_t^+ is empty)

(ii) If v ∈ V_t^0 then G – v is well totally dominated graph .

(iii) If v ∈ V_t^0 then v ∈ W_t^0 .

PROOF :

(i) Ifv ∈ V_t^+ then

$\gamma t(G) < \gamma t(G - v) \leq \Gamma t(G - v) \leq \Gamma t(G)$.

Since$\gamma t(G) = \Gamma t(G)$, this implies that $\gamma t(G - v) = \gamma t(G)$

Which is a contradiction .Thusv ∉ V_t^+.

(ii) If v $\in V_t^0$ then

γt (G) = γt (G - v) $\leq \Gamma t$(G - v) $\leq \Gamma t$ (G).

This impliesthat γt (G - v) =Γt(G - v) .

Thus G – v is well totally dominated graph .

(iii) From (ii) γt (G) = γt (G - v) = Γt(G - v) = Γt(G).

Therefore v $\in W_t^0$. Hence the theorem .■

Next we define so called approximately well totally dominated graphs.

DEFINITION 8

A graph G is said to be an approximately well totally dominated graph , if $\Gamma_t(G) = \gamma_t(G) + 1$.

For example P_5 is an approximately well totally dominated graph .

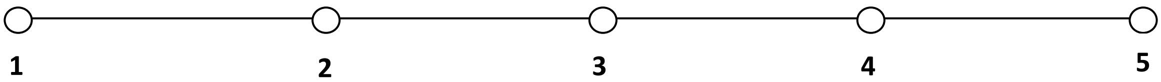

Fig. 2: Path graph with five vertices.

We now prove the following theorem.

THEOREM 6

Let G be a well totally dominated graph and v be a vertex of G such that v $\notin V_t^i$ then either G – v is well totally dominated graph or it is an approximately well totally dominated graph .

PROOF :

Suppose v $\in V_t^-$ therefore

γ_t (G – v) = γ_t (G) – 1 .

γ_t (G – v) $\leq \Gamma_t$(G – v) $\leq \Gamma_t$(G) .

Case : (i) If Γ_t(G – v) = γ_t (G – v) then G – v is well totally dominated graph .

Case : (ii) γ_t (G) $\leq \Gamma_t$(G – v) $\leq \Gamma_t$(G).

Since G is well totally dominated graph , γ_t (G) = Γ_t(G) and thus Γ_t(G – v) = γ_t (G) . Which is equal to γ_t (G – v) + 1. Thus G – v is an approximately well totally dominated graph .

If v $\in V_t^0$, then γ_t (G) = γ_t (G – v) $\leq \Gamma_t$(G – v) $\leq \Gamma_t$(G).

Therefore Γ_t(G – v) = γ_t (G – v) and hence G – v is a well totally dominated graph. ■

THEOREM 7

Suppose G is an approximately well totally dominated graph and v is a vertex such that v $\notin V_t^i$,then if v $\in V_t^+$ then G – v is well totally dominated graph, v $\in W_t^0$ and Γt(G – v) = γt (G) + 1.

PROOF :

Since v$\in V_t^+$,

γ_t (G)<γ_t (G – v)≤ Γ_t (G – v)≤ Γ_t (G)= γ_t (G) + 1.

Therefore γ_t (G – v) = γ_t (G) + 1.

Therefore γ_t (G) + 1 ≤Γ_t(G – v)≤ γ_t (G) + 1.

Hence Γ_t(G – v) = γ_t (G) + 1 = γ_t (G – v).

That is G – v is well totally dominated graph .

Also ,Γ_t(G – v) = γ_t (G) + 1 = Γ_t(G).

Thus v$\in W_t^0$.■

THEOREM 8

Suppose G is an approximately well totally dominated graph and v is a vertex such that v $\notin V_t^i$.

Ifv$\in V_t^0$, then G – v is either an approximately well totally dominatedgraph or it is well totally dominated graph.In the first case v $\in W_t^0$ and in the second case v$\in W_t^-$.

PROOF :

Since v$\in V_t^0$,

γ_t (G)= γ_t (G – v)≤ Γ_t (G – v)≤ Γ_t (G).

Case : (i)

Γ_t(G – v) = Γ_t(G) ,

then Γ_t(G – v) = Γ_t(G)= γ_t (G)+1 = γ_t (G – v)+1.

Thus G – v is an approximately well totally dominated graph.

Since Γ_t(G – v) = Γ_t(G),

That is v $\in W_t^0$.

Case : (ii) Γ_t(G – v) = γ_t (G – v) , then obviously

G – v iswell totally dominated graph .

Since Γ_t(G – v) = γ_t (G – v)= γ_t (G)<Γ_t(G),

Therefore v$\in W_t^-$.■

THEOREM 9

Let G be an approximately well totally dominated graph and v is a vertex such that v $\notin V_t^i$. If v $\in V_t^-$ then exactly one of thefollowing three possibilities holds.

(i) G – v is well totally dominated graph .

(ii) G – v is an approximately well totally dominated graph .

(iii) v $\in W_t^0$.

PROOF :

Since v $\in V_t^-$,

γ_t (G – v) = γ_t (G) – 1.

Now ,γ_t (G – v)≤ Γ_t (G – v)≤ Γ_t (G).

Therefore Γ_t(G – v)= γ_t (G) – 1 , or Γ_t(G – v)= γ_t (G) or Γ_t(G – v)=Γ_t(G).

If the first equality holds then Γ_t(G – v) = γ_t (G – v) and the graph G – v is well totally dominated graph.

If the second equality is true then Γ_t(G – v)= γ_t (G)= γ_t (G – v)+1 , and so the graph is an approximately well totally dominated graph .If the third equality holds then v $\in W_t^0$. ■

REFERENCES

[1] D.K.Thakkar andG.J.vala. Graphs critical with respect to the total domination, Mathematics Today, Volume 26, December,2010,1-13.

[2] J.Topp and Volkmann. Well covered and well dominated block graphs and unicyclic graphs, Math., Pannon, 1:55-66,1990.

[3] M.Plummer. Well-covered graphs: a survey, Questions Math.,16:253-287,1993.

[4] Teresa W. Hynes, Stephen T. Hedetniemi, Peter J. Slater.FundamentalsofDomination in Graphs, 1998, Marcel Dekkar,Inc.

[5] Teresa W. Hynes, Stephen T. Hedetniemi, Peter J. Slater.Domination in Graphs(Advanced Topics), 1998, Marcel Dekkar, Inc.