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The Longitudinal Dispersion of Miscible Fluids Through Porous Media

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Abstract

The present paper discusses the phenomenon of longitudinal dispersion which is the process by which miscible fluids in laminar flow mix in the direction of the flow. The phenomenon is discussed by regarding the cross-sectional flow velocity as constant. The mathematical formulation leads to a non – linear partial differential equation which is transferred into linear partial differential equation by using similarity transformation. A Fourier sine series solution has been obtained in terms of exponential form.

Keywords: Miscible displacement, Longitudinal Dispersion, Porous Media.

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1.1 INTRODUCTION:

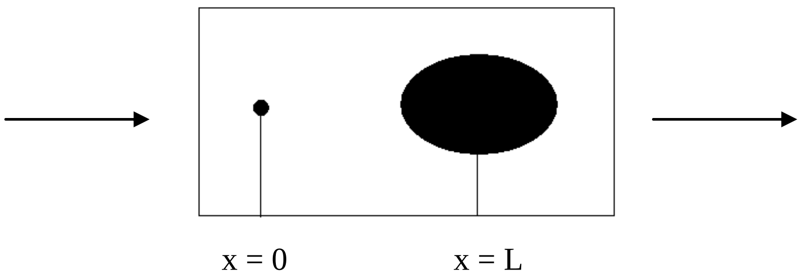
This paper deals with the discussion of miscible displacement. This type of displacement has been of great concern to hydrologists who have been studying the problem of displacement of fresh water by seawater in coastal areas. Recently a modern note has been struck as the problem has become important to people who are trying to dispose safely underground of ever-increasing amounts of atomic waste products from nuclear reactors. The oil industry has also become involved in miscible displacement studies in connection with the possibility of flushing oil by solvents from reservoirs.

Many researchers have discussed this phenomenon from different aspects : for example carrier [2] , Bear [1] , Greenkorn [3], Raval [5], Patel & Mehta [4]. This paper discusses the phenomenon of longitudinal dispersion which is the process by which miscible fluids in laminar flow mix in the direction of the flow. The phenomenon is discussed by regarding the cross-sectional flow velocity as constant. The mathematical formulation leads to a non – linear partial differential equation which is transferred into linear partial differential equation by using similarity transformation. A Fourier sine series solution has been obtained in terms of exponential form.

1.2 STATEMENT OF THE PROBLEM :

Miscible displacement in porous media is a type of double – phase flow in which the two phases are completely soluble in each other. Therefore, capillary forces between the two fluids do not come into effect. At first it must be thought that miscible displacement could be described in a very simple fashion. The mixture under conditions of complete miscibility, could be thought to behave, locally at least, as a single phase fluid which would obey Darcy's law. The change of concentration, in turn, would be caused by diffusion along the flow channels and thus be governed by the bulk coefficient of diffusion of the one fluid in the other. In this fashion, one arrives at a heuristic description of miscible displacement which looks at a first glance, at least very plausible.

The problem is to describe the growth of the mixed region, i.e. to find concentration as a function of time t and position x, as the two miscible fluids flow through porous media. Outside of the mixed zone (on either side) the single – fluid equation describes the motion. The problem is more complicated, even in one dimension with fluids of equal properties, since the mixing takes place both longitudinally and transversely. We inject a "dot" of traced fluid of concentration C_1. This situation is sketched in the following figure.



Longitudinal and Transverse Dispersion ([4])

As the dot moves from left to right it will spread in the direction of flow and perpendicular to the flow. At the right the dot has transformed into an ellipse with concentration varying from C_1 to C_2 across it.

1.3 MATHEMATICAL FORMULATION:

According to Darcy's law, the equation of continuity for the mixture, in the case of incompressible fluid is given by,

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho \bar{V}\right) = 0 \qquad (1.3.1)$$

where ρ is the density for mixture and $\bar{V}$ is the pore seepage velocity vector.

The equation of diffusion for a fluid flow through a homogeneous porous medium, without increasing or decreasing the dispersing material is given by :

$$\frac{\partial C}{\partial t} + \nabla.\left(C\bar{V}\right) = \nabla.\left[\rho \bar{D} \nabla \left(\frac{C}{\rho}\right)\right] \qquad (1.3.2)$$

where C is the concentration of the fluid A into the other host fluid B and D is the tensor of co-efficients of dispersion with nine components.

In a laminar flow through a homogeneous porous medium at constant temperature ρ is constant. Then,

$$\nabla.\bar{V} = 0 \tag{1.3.3}$$

and equation (1.3.2) becomes

$$\frac{\partial C}{\partial t} + \bar{V}.\nabla C = \nabla\left(\bar{D}\nabla C\right) \tag{1.3.4}$$

Since the porous media is considered as homogeneous, the seepage velocity $\bar{V}$ along x - axis will be constant over the cross section and $\bar{D}$ is expressed as

$$\bar{D} = D_{11}\bar{i} + D_{22}\bar{j} + 0$$

where $D_{11} = D_L$ (Coefficient of longitudinal dispersion)
and $D_{22} = D_T$ (Coefficient of transverse dispersion)
and all other D_{ij} are zero. In this case eq. (1.3.4) becomes,

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D_L\frac{\partial^2 C}{\partial x^2} \tag{1.3.5}$$

where u is the component of constant flow velocity along x-axis and $D_L > 0$

The boundary conditions in longitudinal dispersion are

$$C(0,t) = C_1(t),\ (t > 0) \tag{1.3.6}$$

And $C(L,t) = C_2(t),\ (t > 0)$ (1.3.7)

C_1 is the concentration of the tracer (one fluid A) at $x = 0$ and C_2 is the concentration of the tracer (of the same fluid) at $x = L$

1.4 SOLUTION OF THE PROBLEM

Using the transformation given in [4], $\xi = x - ut, \tau = t$

Equation (1.3.5) will reduce to the form

$$\frac{\partial C}{\partial t} = D_L\frac{\partial^2 C}{\partial \xi^2} \tag{1.4.1}$$

and the conditions (1.3.6), (1.3.7) are redefined as

$$C(-u\tau,\tau)=C_1(\tau) \tag{1.4.2}$$

and $C(L-u\tau,\tau)=C_2(\tau)$ (1.4.3)

Equation (1.4.1) together with the conditions (1.4.2) and (1.4.3) represents boundary value problem.

As the porous media is homogeneous unsaturated, after initial concentration there will be transient change in concentration and after some time it becomes steady under the interconnected capillary pressure effect.Therefore, $C(\xi,\tau)$ can be expressed as the sum of transient and steady state concentration.

Hence,

$$C(\xi\ \tau)=C_s(\xi)+C_t(\xi,\tau) \tag{1.4.4}$$

For steady state concentration, $\frac{\partial^2 C_s}{\partial \xi^2}=0$

whose general solution is, $C_s(\xi)=A\xi+B$

when $\xi = 0$, $C_{(s)}(0,\tau)=C_1(\tau)$ implying $B=C_1$

Using $C_s(L,t)=C_2(\tau)$, we have $A=\frac{C_2-C_1}{L}$

Hence, the steady state concentration is

$$C_{(s)}(\xi)=\left(\frac{C_2-C_1}{L}\right)\xi+C_1 \tag{1.4.5}$$

For the transient concentration, the boundary conditions are redefined as

$$C_t(0,\tau)=C(0,\tau)-C_s(0)=C_1-C_1=0 \tag{1.4.6}$$

$$C_t(L,\tau)=C(L,\tau)-C_s(L)=C_2-C_2=0 \tag{1.4.7}$$

We assume that $C_t(\xi,0)=f(\xi), 0\le\xi\le L$ (1.4.8)

Thus for the transient concentration, we have to solve the given PDE (1.1) subject to the conditions (1.4.6) and (1.4.7).

The acceptable solution is given by the equation

$$C_t(\xi,\tau)=\{P\cos\lambda\xi+Q\sin\lambda\xi\}\left\{A\,e^{-\lambda^2 D_L\tau}\right\}$$

Using condition (1.4.6) and (1.4.7) we have,

$$0=P\left(Ae^{-\lambda^2 D_L\tau}\right)\text{and}$$

$$0=\{P\cos\lambda L+Q\sin\lambda L\}\left(Ae^{-\lambda^2 D_L\tau}\right)$$

Therefore, $P=0$, and

$\sin\lambda L=0\Rightarrow\lambda=\frac{n\pi}{L}$, where n is an integer.

Therefore,

$$C_t(\xi,\tau)=A_n e^{\frac{-n^2\pi^2 D_L\tau}{L^2}}\sin\left(\frac{n\pi\xi}{L}\right)$$

General solution is ,

$$C_t(\xi,\tau)=\sum_{n=0}A_n e^{\frac{-n^2\pi^2 D_L\tau}{L^2}}\sin\left(\frac{n\pi\xi}{L}\right) \tag{1.4.9}$$

Using condition (1.4.8) we have

$$f(\xi)=\sum_{n=0}A_n Sin\left(\frac{n\pi\xi}{L}\right),\qquad 0\le\xi\le L \tag{1.4.10}$$

Therefore,

$$\int_0^L f(\xi)\sin\frac{m\pi\xi}{L}\,d\xi=\sum A_n\int_0^L\sin\frac{m\pi\xi}{L}\sin\frac{n\pi\xi}{L}\,d\xi \tag{1.4.11}$$

where m is also an integer. Using

$$\int_0^L\sin\frac{m\pi\xi}{L}\sin\frac{n\pi\xi}{L}\,d\xi=\begin{cases}0, & m\neq n\\ \frac{L}{2}, & m=n\end{cases}$$

From equation (1.4.11), we have,

$$A_m = \frac{2}{L}\int_0^L f(V)\sin\left[\frac{m\pi V}{L}\right]dV \tag{1.4.12}$$

$$C_t(\xi,\tau) = \sum A_n e^{\left(-n^2\pi^2 D_L\tau\right)/L^2} \sin\left(\frac{n\pi\xi}{L}\right) \tag{1.4.13}$$

Hence, the desired analytical expression for the concentration distribution at any time τ is given by,

$$C(\xi,\tau) = \left(\frac{C_2 - C_1}{L}\right)\xi + C_1 + \sum A_n e^{\frac{\left(-n^2\pi^2 D_L\tau\right)}{L^2}} .\sin\left(\frac{n\pi\xi}{L}\right) \tag{1.4.14}$$

Since $\xi = x - ut,\ \tau =$ t, equation (1.4.14) became,

$$C(x,t) = \left(\frac{C_2 - C_1}{L}\right)(x - ut) + C_1 + \sum A_n e^{\left(\frac{-n^2\pi D_L t}{L^2}\right)} .\sin\left(\frac{n\pi(x - ut)}{L}\right) \tag{1.4.15}$$

1.6 CONCLUDING REMARKS

Solution (1.4.15) represents concentration distribution at any time t and position x. The solution obtained is a half range Fourier sine series which converges satisfactorily for large and moderate values of 'n' in preference to error function type of solution. Then it can be used in numerical evaluation.

The equation (1.4.15) is analytical solution of the problem. We have not included numerical as well as graphical representation due to our particular interest for analytical expression.

1.7 APPLICATIONS

The dispersion theory has been applied to the explanation of various physical phenomena and to the prediction of events in practical engineering applications as follows :

(a) To the mixing of ground water and sea water in permeable subsoils. This is probably one of the oldest applications of miscible displacement theory. Originally, only the elementary displacement theory discussed in [5]. A modern application of dispersion theory to this case has been discussed by Carriere [2] who solved the diffusivity equation for the linear case and made some estimates and predictions with regards to the salinity distribution on Maui Island, Hawaii.

(b) To the hydrodynamic dispersion of solutes in the soil moisture stream, which is a problem related to that of salt-water intrusion. It has now become of great importance in connection with the problem of dispersion of atomic wastes.

(c) To the idea of recovering oil by solvent floods from underground reservoirs. This has been discussed in a great number of papers in the literature; reviews of the prevailing ideas may be given.

Reviewing the above practical cases it may therefore be said that the dispersion theory provides a satisfactory description of miscible displacement phenomena in porous media.

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