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Non-homeomorphic sequential fan like spaces

N. P. Shrimali* and D. K. Thakkar**

*Department of Mathematics, Gujarat University, Ahmedabad, Gujarat

E-mail: $narenp05@yahoo.co.in$

**Department of Mathematics, Saurashtra University, Rajkot, Gujarat

E-mail: $dkthakkar@yahoo.co.in$

Abstract

Sequential fan is an example of a countable, T_1, hemicompact, non-locally-compact, Fréchet-Urysohn space having exactly one limit point which is not first countable [2]. In this paper we discuss some topological spaces which can be constructed like sequential fan which are not homeomorphic to sequential fan. Two of these spaces turn out to be homeomorphic.

Keywords: First countable, Fréchet-Urysohn, Sequential fan, Hemicompact.

AMS subject classification (2010): 54A20, 54D55.

1 Introduction

The space called Sequential fan got wide publicity by well known papers of S. P. Franklin entitled as spaces in which sequences suffice - I & II [4],[5]. Then after several people as for example Frank Siwiec, A. K. Desai have used sequential fan and constructed spaces like it [1],[2]. Here we discuss examples which can be constructed like sequential fan but not homeomorphic to it.

2 Definitions

Definition 2.1. A space X is said to have a *countable basis* at x if there is a countable collection $\mathscr{B}$ of neighborhoods of x (open sets containing x) such that each neighborhood of x contains at least one of the elements of $\mathscr{B}$. A space that has a countable basis at each of its points is said to satisfy the *first countability axiom*, or to be *first countable*. Note that every metrizable space satisfies this axiom.

Definition 2.2. A topological space X is called a *Fréchet-Urysohn space (or Fréchet space)* if for every $A \subseteq X$ and every $x \in \bar{A}$ there exists a sequence (x_n) of points of A converging to x.
Note that every metric space is Fréchet-Urysohn. More generally, any first countable space is Fréchet-Urysohn.

Definition 2.3. A topological space X is said to be *hemicompact* if X has a countable cover $\mathscr{C}$ of compact subspaces such that if K is a compact subset of X then there exists a $C \in \mathscr{C}$ for which $K \subseteq C$.

3 Sequential Fan

Consider countably many disjoint copies of a convergent sequence (i.e. copies of $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ as subsets of real line) and identify the limit points, denote this new identified point by 0 and the resulting space by the set X. New space X is called *sequential fan*.
Some authors have used different notations for sequential fan. For example, F was used by S.P.Franklin and M. Rajagopalan [3], $S(w)$ by Shou Lin [6], and S_ω by Yoshio Tanaka [7].
The quotient topology on X is described in the following way :

1. Any subset of X not containing 0 is open.
2. Suppose $0 \in U \subseteq X$, then U is open if and only if U contains all but finitely many points of each copy of the convergent sequence.

The sequential fan has following properties:

(i) Countable

(ii) Not first countable

(iii) Fréchet-Urysohn

(iv) Hemicompact

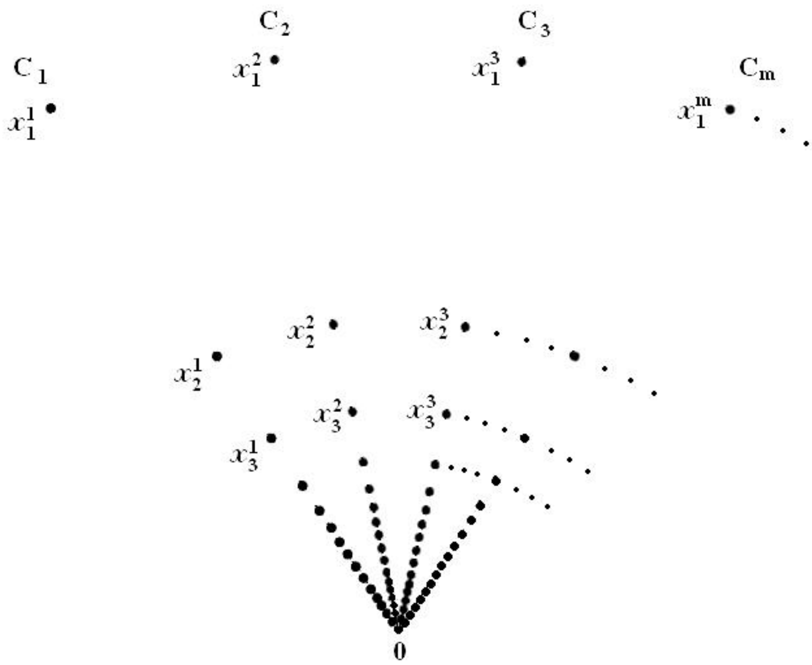
(v) Non-locally-compact

(vi) Having exactly one limit point

We can write sequential fan X as a set in the following way :

$$X = \left(\bigcup_{j=1}^{\infty} C_j \right) \cup \{0\},$$

where C_j is the set of points of convergent sequence. (That is, $C_j = \{x_i^j | i = 1, 2, ...\}$). So we can visualize the sequential fan as in the following picture.



X

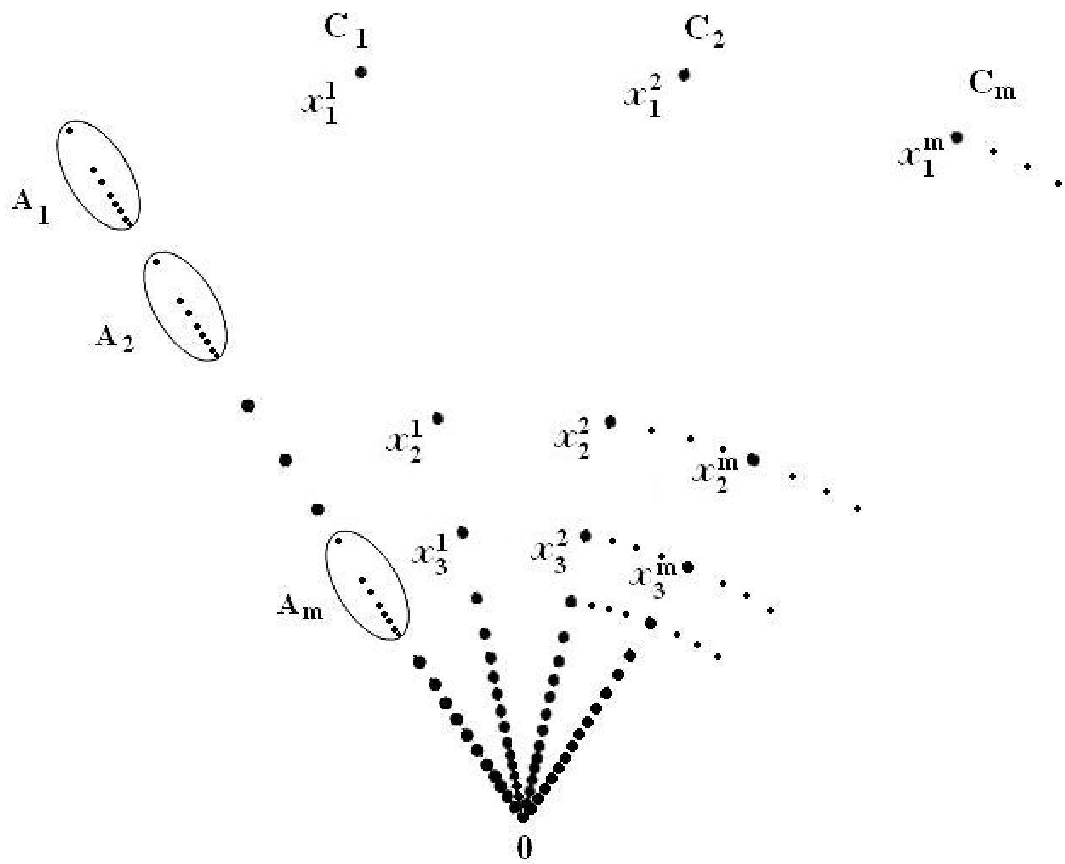
[Note that sequential fan X is not a subspace of $\mathbb{R}^2$]

4 Examples X_1, X_2 and X_3

(1) Let $X_1 = \{0\} \cup \mathbb{N} = \{0\} \cup \left(\bigcup_{j=1}^{\infty} C_j\right) \cup \left(\bigcup_{i=1}^{\infty} A_i\right)$ where $\{A_i, C_j | i = 1, 2, ..., \quad j = 1, 2, ...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$. The topology τ_1 on X_1 is defined as follows:

(a) Any subset of X_1 not containing 0 is open.

(b) suppose $0 \in U \subseteq X_1$, then U is open in X_1 if and only if U contains all but finitely many points from each C_j and all but finitely many A_is.
So we can visualize this space in the following picture:



X_1

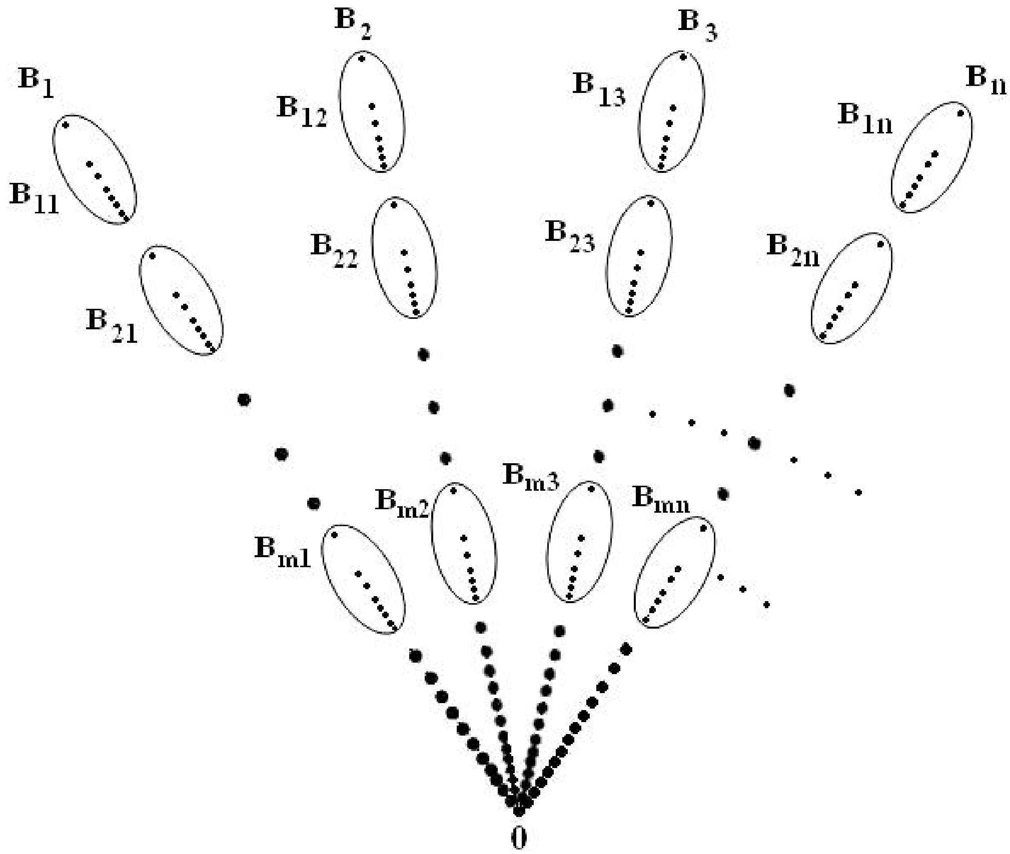
[Note that the space X_1 is not a subspace of $\mathbb{R}^2$]

(2) Let $X_2 = \{0\} \cup \mathbb{N} = \{0\} \cup \left(\bigcup_{j=1}^{\infty} B_j\right)$, and $B_j = \bigcup_{i=1}^{\infty} B_{ij}$ where $\{B_{ij} | i, j = 1, 2, ...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$. The topology τ_2 on X_2 is defined as follows:

(a) Any subset of X_2 not containing 0 is open.

(b) Suppose $0 \in U \subseteq X_2$, then U is open in X_2 if and only if U contains all but finitely many B_{ij}s from each B_j.

So we can visualize this space in the following picture:



X_2

[Note that the space X_2 is not a subspace of $\mathbb{R}^2$]

(3) Let $X_3 = \{0\} \cup \mathbb{N}$

$$= \{0\} \cup \left(\bigcup_{j=1}^{\infty} B_j \right) \cup \left(\bigcup_{j=1}^{\infty} C_j \right)$$

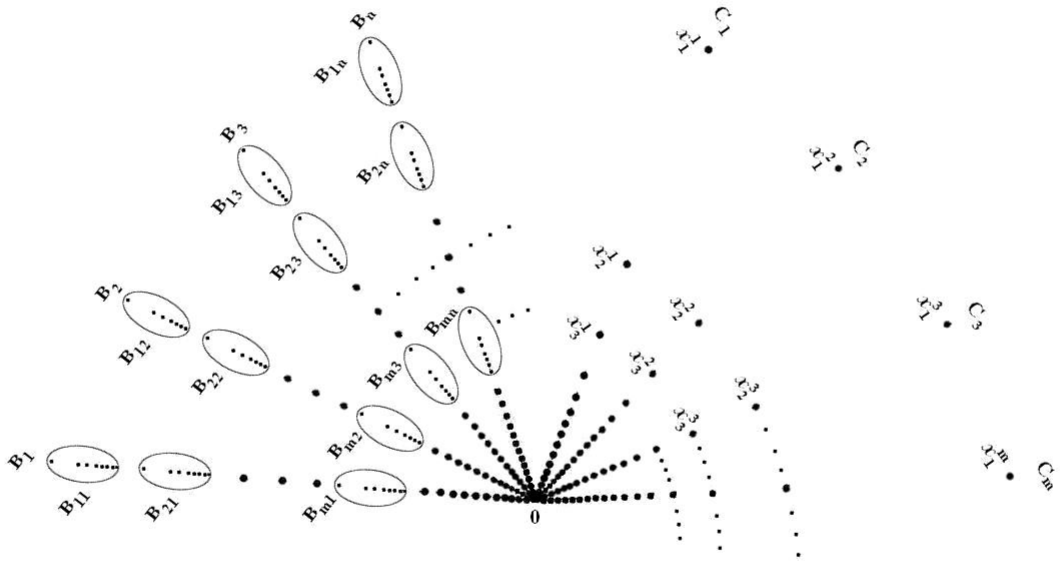
Where $B_j = \bigcup_{i=1}^{\infty} B_{ij}$ and $\{B_{ij}, C_j | i = 1, 2, ..., \ j = 1, 2, ...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$.

The topology τ_3 on X_3 is defined as follows :

(i) any subset X_3 not containing 0 is open.

(ii) Suppose $0 \in U \subseteq X_3$, then U is open in X_3 if and only if U contains all but finitely many points from each C_j and all but finitely many B_{ij}s from each B_j.

So we can visualize this space in the following picture:



X_3

[Note that the space X_3 is not a subspace of $\mathbb{R}^2$]

5 Are all these four spaces different ?

Theorem 5.1. *The spaces X_1 and X_2 are not homeomorphic to sequential fan X.*

Proof:

(1) The space X_1 is not homeomorphic to sequential fan X:

Suppose f is a homeomorphism from X_1 to X.

Claim: $f(A_m) \cap C_i$ is finite for all i and for all m.

Suppose $f(A_k) \cap C_j$ is infinite for some j and for some k. Then the points of this set form a sequence (y_n) in X which converges to 0 and since f is a homeomorphism we have a sequence (x_n), where $f(x_n) = y_n$ and $x_n \in A_k$, in X_1 must converge to 0. But no sequence in A_k converges to 0, which is a contradiction. Thus our claim is proved.

Let j_1 be the first index such that $f(A_1) \cap C_{j_1}$ is nonempty. Choose a point y_1 in $f(A_1) \cap C_{j_1}$. Let j_2 be the first index after $j_1 (i.e.\ j_2 > j_1)$ such that $f(A_2) \cap C_{j_2}$ is nonempty. Choose a point y_2 in $f(A_2) \cap C_{j_2}$. Continuing in this way we get a sequence (y_n) in X. Since f is onto, corresponding to each y_n there exists x_n in A_n such that $f(x_n) = y_n$. Thus we get a sequence (x_n) in X_1.

Claim: The sequence (x_n) converges to 0 in X_1.

Let U be any open set containing 0 in X_1. Then $\exists$ a positive integer N such that $\bigcup_{n\geq N} A_n \subseteq U$. Thus $\forall n \geq N, x_n \in U$. This proves our claim.

Since f is a homeomorphism, the sequence (y_n) must converge to $f(0) = 0$. But $U = X\backslash\{y_1, y_2, ...\}$ is an open set containing 0 in X which does not contain any point of the sequence (y_n). So the sequence (y_n) cannot converge to 0. Which is a contradiction to the fact that f is a homeomorphism. Thus our supposition is false and we conclude that X_1 is not homeomorphic to X.

(2) The space X_2 is not homeomorphic to sequential fan X:

Suppose f is a homeomorphism from X_2 to X.

Claim: $f(B_{ij}) \cap C_k$ is finite for all i, j, k.

Suppose that $f(B_{mn}) \cap C_t$ is infinite for some m, for some n and for some t. Then the points of this set form a sequence (y_n) in X which converges to 0 and since f is a homeomorphism we have a sequence (x_n), where $f(x_n) = y_n$ and $x_n \in B_{mn}$, in X_2 must converge to 0. But no sequence in B_{mn} converges to 0 in X_2, which is a contradiction. This proves our claim.

Let j_1 be the first index such that $f(B_{11}) \cap C_{j_1}$ is nonempty. Choose a point y_1 in $f(B_{11}) \cap C_{j_1}$. Let j_2 be the first index after j_1 (i.e. $j_2 > j_1$) such that $f(B_{21}) \cap C_{j_2}$ is nonempty. Choose a point y_2 in $f(B_{21}) \cap C_{j_2}$. Continuing in this way, we get a sequence (y_n) in X. Since f is onto, corresponding to each y_n there exists x_n in B_{n1} such that $f(x_n) = y_n$. Thus we get a sequence (x_n) in X_2.

Claim The sequence (x_n) converges to 0 in X_2.

Let U be any open set containing 0. Then $\exists$ a positive integer N such that $\bigcup_{n\geq N} B_{n1} \subseteq U$. Thus $\forall n \geq N, x_n \in U$ and hence (x_n) converges to 0.

Since f is a homeomorphism, the sequence (y_n) must converge to $f(0) = 0$. But $U = X\backslash\{y_1, y_2, ...\}$ is an open set containing 0 in X which does not contain any point of the sequence (y_n). So the sequence (y_n) cannot converge to 0. Which is a contradiction to the fact that f is a homeomorphism. This shows that X_2 is not homeomorphic to X.

Theorem 5.2. *The spaces X_1 and X_2 are not homeomorphic.*

Proof: Suppose that f is a homeomorphism from X_1 to X_2.

<u>Claim</u>: $f(A) \not\subseteq \bigcup_{i=1}^{n} B_i$ for any n, where $A = \bigcup_{i=1}^{\infty} A_i$.

Suppose $f(A) \subseteq \bigcup_{i=1}^{k} B_i$ for some k. Consider B_{k+1}. We assert that $f^{-1}(B_{i(k+1)}) \cap C_j$ is finite for all i and for all j. Suppose $f^{-1}(B_{m(k+1)}) \cap C_t$ is infinite for some t and for some m. Then the points of $f^{-1}(B_{m(k+1)}) \cap C_t$ form a sequence (x_n) in C_t which converges to 0 in X_1 and hence whose image sequence $(f(x_n))$ which is a sequence in $B_{m(k+1)}$ must converge to 0 in X_2. Which is a contradiction to the fact that no sequence in $B_{m(k+1)}$ converges to 0 in X_2. Thus our assertion is proved.

Let j_1 be the first index such that at least one point say x_1 in C_{j_1} whose image $f(x_1)$ lies in $B_{1(k+1)}$. This is possible because $f^{-1}(B_{1(k+1)}) \cap C_j$ is finite for all j. Similarly, let j_2 be the first index after j_1 ($i.e.$ $j_2 > j_1$) such that at least one point say x_2 in C_{j_2} whose image $f(x_2)$ lies in $B_{2(k+1)}$. Inductively, we get a sequence (x_n) in X_1 where $x_n \in f^{-1}(B_{n(k+1)}) \cap C_{j_n}$ $(j_1 < j_2 < ...)$ and a sequence $f(x_n)$ in X_2. The sequence (x_n) does not converge to 0 in X_1 because $X_1 \backslash \{x_1, x_2, ...\}$ is an open set containing 0 in X_1 which does not contain any point of sequence (x_n). But the sequence $f(x_n)$ converges to 0 in X_2 because for any open set U containing 0, $\exists$ a positive integer N such that $\forall n \geq N$, $\bigcup_{n \geq N} B_{n(k+1)} \subseteq U$ and hence $\forall n \geq N, f(x_n) \in U$. Thus, we get a contradiction to the fact that f is a homeomorphism. Hence our claim is proved.

Let j_1 be the first index such that $f(A_1) \cap B_{j_1} \neq \phi$. Therefore $f(A_1) \cap B_{i_1 j_1} \neq \phi$ for some i_1 ($\because$ $B_{j_1} = \bigcup_{i=1}^{\infty} B_{ij_1}$). Choose a point $y_1 \in f(A_1) \cap B_{i_1 j_1}$. Then $y_1 = f(x_1)$ for some $x_1 \in A_1$. Put $A^1 = A \backslash A_1$. By the same argument given above we can prove that $f(A^1) \not\subseteq \bigcup_{i=1}^{n} B_i$ for any n. Then $f(A^1) \cap B_j \neq \phi$ for some $j > j_1$. Let j_2 be the first index after j_1 ($i.e.$ $j_2 > j_1$) such that $f(A^1) \cap B_{j_2} \neq \phi$. Therefore $f(A^1) \cap B_{i_2 j_2} \neq \phi$ for some i_2 ($\because$ $B_{j_2} = \bigcup_{i=1}^{\infty} B_{ij_2}$). Choose $y_2 \in f(A^1) \cap B_{i_2 j_2}$ for some i_2. Then for some k, there exists a point $x_2 \in A_k$ $(k \neq 1)$ such that $f(x_2) = y_2$. Put $A^2 = A \backslash (\bigcup_{i=1}^{k} A_i)$. By the same argument given above we can prove that $f(A^2) \not\subseteq \bigcup_{i=1}^{n} B_i$ for any n. Then $f(A^2) \cap B_j \neq \phi$ for some $j > j_2$. Let j_3 be the first index after j_2 (i.e. $j_3 > j_2$) such that $f(A^2) \cap B_{j_3} \neq \phi$. Therefore $f(A^2) \cap B_{i_3 j_3} \neq \phi$ for some i_3 ($\because$ $B_{j_3} = \bigcup_{i=1}^{\infty} B_{ij_3}$). Choose a point $y_3 \in f(A^2) \cap B_{i_3 j_3}$. Then for some t, where $t \notin \{1, 2, ...k\}$, there exists a point $x_3 \in A_t$ such that $f(x_3) = y_3$. Inductively we get a sequence (x_n) in X_1 and a sequence (y_n) in X_2.

(x_n) converges to 0:

Let U be any open set containing 0. Then $\exists$ a positive integer N such that $\bigcup_{n\geq N} A_n \subseteq U$. Thus $\forall n \geq N, x_n \in U$.

(y_n) does not converge to 0:

$U = X_2 \backslash (\bigcup_{n=1}^{\infty} B_{i_n j_n})$ is an open set containing 0 in X_2 which does not contain any point of sequence (y_n).

Thus the sequence (x_n) converges to 0 in X_1 whereas (y_n) does not converge to 0 in X_2. Which is a contradiction to the fact that f is a homeomorphism. Thus our supposition that X_1 and X_2 are homeomorphic is wrong. Hence the theorem.

Remark: Equivalently we can prove that X_1 and X_2 are not hemicompact and hence they are not homeomorphic to X (sequential fan).

Theorem 5.3. *The spaces X_2 and X_3 are homeomorphic.*

Proof: We have

$$X_2 = \{0\} \cup \mathbb{N} = \{0\} \cup \left(\bigcup_{j=1}^{\infty} B_j \right),$$

where $B_j = \bigcup_{j=1}^{\infty} B_{ij}$ and $\{B_{ij} | i, j = 1, 2, ...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$.

And

$$X_3 = \{0\} \cup \mathbb{N} = \{0\} \cup \left(\bigcup_{j=1}^{\infty} B_j \right) \cup \left(\bigcup_{j=1}^{\infty} C_j \right)$$

where $B_j = \bigcup_{i=1}^{\infty} B_{ij}$ and $\{B_{ij}, C_j | i = 1, 2, ..., j = 1, 2, ...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$.

Since B_{ij} and C_j are countable we can write B_{ij} and C_j as follows:

$$\begin{aligned} B_{ij} &= \{x^i_{kj} \mid k = 1, 2, ...\} \;\; i, j = 1, 2, ... \\ \text{and } C_j &= \{x^j_k \mid k = 1, 2, ...\} \;\; j = 1, 2, ... \end{aligned}$$

We define a map f from X_3 to X_2 as follows:

$$\begin{aligned} f(x^j_k) &= x^k_{1j} \\ f(x^i_{kj}) &= x^i_{(k+1)j} \\ f(0) &= 0 \end{aligned}$$

(1) Clearly, f is one-one and onto.

(2) f is continuous:

Let U be any open set in X_2. If $0 \notin U$, then $0 \notin f^{-1}(U)$. Therefore $f^{-1}(U)$ is open in X_3. If $0 \in U$, then U contains all but finitely many B_{ij} from each B_j, say,
$B_{11}, B_{21}, ..., B_{\alpha_1 1}, B_{12}, B_{22}, ..., B_{\alpha_2 2}, ..., B_{1j}, B_{2j}, ..., B_{\alpha_j j}, ...$
Then $f^{-1}(U)$ contains 0 and all B_{ij} except
$B_{11}, ..., B_{\alpha_1 1}, B_{12}, B_{22}, ..., B_{\alpha_2 2}, ..., B_{1j}, ..., B_{\alpha_j j}, ...$ and contain all points except finitely many from each C_j which are
$x_1^1, ..., x_{\alpha_1}^1, x_1^2, ..., x_{\alpha_2}^2, ..., x_1^j, ..., x_{\alpha_j}^j....$
Therefore $f^{-1}(U)$ is open in X_3. Hence f is continuous.

(3) f is an open map:

Let U be any open set in X_3. If $0 \notin U$, then $0 \notin f(U)$. Therefore $f(U)$ is open in X_2. If $0 \in U$, then U contains all but finitely many B_{ij} from each B_j, say, $B_{11}, ..., B_{\alpha_1 1}, B_{12}, ..., B_{\alpha_2 2}, ..., B_{1j}, ..., B_{\alpha_j j}, ...$ and contains all but finitely many points from each C_j, say, $x_1^1, ..., x_{\beta_1}^1, x_1^2, ..., x_{\beta_2}^2, ..., x_1^j, ..., x_{\beta_j}^j, ...$ Without loss of generality we assume $\beta_i \leq \alpha_i$. Then $f(U)$ contains all B_{ij} from each B_j except $B_{11}, ..., B_{\alpha_1 1}, B_{21}, ..., B_{\alpha_2 2}, ..., B_{1j}, ..., B_{\alpha_j j}, ...$ Thus $f(U)$ is open in X_2. Hence f is open.

From (1), (2) and (3), f is a homeomorphism. That is, X_2 and X_3 are homeomorphic.

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References

[1] Desai, A. K., Some peculiar countable spaces construction and characterization, Ph.D. Thesis Gujarat University (1982)

[2] Frank Siwiec, countable spaces having exactly one nonisolated point. I., Proc.AMS.97(2)354-350 June(1976)

[3] Franklin, S. P. and Rajagopalan, M., Some examples in topology, Trans. Amer. Math. Soc., Vol.155, No.2 (1971), 305-314. MR 44#972.

[4] Franklin, S. P., Spaces in which sequences suffice, Fund. Math. 57 (1965), 107-115. MR 31#5184.

[5] Franklin, S. P., Spaces in which sequences suffice II, Fund. Math. 61 (1967), 51-56. MR 36#5882.

[6] Lin, S., A note on the Arens′ space and sequential fan, Topology and its Applications 81 (1997),185-196. MR1485766 (98m:54008).

[7] Tanaka, Y., Metrizability of GO-spaces and topological groups, Bulletin of Tokyo Gakugei Uni. Sect. IV, 53 (2001), 7-18. MR 1875367 (2002k:54034).