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Weak-type estimate for maximal functions associated to real-analytic functions

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Abstract

Let $\phi(s,t) = \sum a_{m,n} s^m t^n$ be a real-analytic function in a neighbourhood of $(0,0) \in R^2$ such that $\phi(0,0) = \nabla\phi(0,0) = 0$ and suppose that the series converges for all $|s|, |t| \le 4$. Consider the maximal function

$$\mathcal{M}f(x) = \sup_{0<h,k<c} \frac{1}{hk} \left| \int_0^h \int_0^k f(x - \phi(s,t)) \mathrm{d}s\mathrm{d}t \right|,$$

where c is a sufficiently small positive number. In [5] we have shown that if ϕ is a polynomial function then $\mathcal{M}$ is weak-type 1-1. In this paper we extend the result of [5] to the case of real-analytic functions under certain assumptions.

Keywords: Maximal functions, weak-type 1-1 bound, Newton diagram.

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1 Introduction

Over the last decade or so, a lot of focus is on the study of L^p and the weak-type estimates of the multiple Hilbert transform (and related maximal functions), which are associated to polynomial or real-analytic functions. See for instance [1], [2], [3], [5], [6], [7], [8], and [9]. This paper adds one more result into that category. For convenience, we shall follow the notations and terminology used in [5] as much as we can. We define the Newton diagram (denoted by Π) associated to the function ϕ as the smallest closed convex set containing

$$\bigcup_{(m,n)\in\Lambda} \{(x,y) \in R^2 | x \ge m, y \ge n\}$$

where $\Lambda = \{(m,n) \in Z^+ \times Z^+ : a_{m,n} \ne 0\}$. Clearly Π is an unbounded polygon with a finite number of corners (vertices) and if $\mathcal{D}$ denote the set of its vertices then $\mathcal{D} \subseteq \Lambda$. Furthermore, suppose $\mathcal{D}$ consists of r vertices $v_1, v_2, \ldots, v_r$ with $v_j = (m_j, n_j)$ for $1 \le j \le r$. We choose the order of the points so that m_{j+1} is strictly greater than m_j and consequently n_{j+1} is strictly less than n_j. With these terminology, our result reads as follows.

Theorem 1.1 *$\mathcal{M}$ is weak-type 1-1 if the set Λ has at the most one point on X-axis and at the most one point on Y-axis.*

Remark: Λ has at the most one point on X-axis or Y-axis, also means that the real-analytic function $\phi(s,t)$ has at the most one term of the type s^m and at the most one term of the type t^n. Thus if $\phi(s,t)$ does not contain pure term of the type s^m or t^n, then Theorem 1.1 guarantees weak-type 1-1 estimate for $\mathcal{M}$.
The aim of this paper is to discuss the modifications in the proof of [5] so that it applies to the real-analytic case as well. We shall avoid the details where the arguments seem to be repeating. In order to discuss these modifications, we shall need a few more notations and terminolgy related to the Newton diagram Π. So we first do this.
For $r \geq 2$ and $1 \leq j \leq r-1$, let $\overline{n}_j = (n_j^1, n_j^2)$ denote an inward normal vector to the edge $\overline{v_j v}_{j+1}$ such that n_j^1 and n_j^2 are positive integers. Here the choice is not unique but it does not matter what we choose. We also let $\overline{n}_0$ and $\overline{n}_r$ denote the the unit vectors $(1,0)$ and $(0,1)$ respectively. Note that the convexity of Π immediately implies that

$$\overline{n}_j \cdot (v - v_j) \geq 0 \text{ and } \overline{n}_{j-1} \cdot (v - v_j) \geq 0 \tag{1}$$

for all $v \in \Lambda$ and $1 \leq j \leq r$. Further if $r \geq 2$ and $1 \leq j \leq r-1$, let θ_j denote the angle of $\overline{n}_j$ with the positive X-axis and let

$$\rho_j = \min\{\cos(\theta_j), \cos(\frac{\pi}{2} - \theta_j)\}.$$

Clearly $0 < \theta_j < \pi/2$, and so $\rho_j > 0$ for $1 \leq j \leq r-1$. Now for $1 \leq j \leq r$, let ϵ_j denote the angle between $\overline{n}_{j-1}$ and $\overline{n}_j$ and further let

$$\kappa_j = \cos(\frac{\pi}{2} - \epsilon_j).$$

Since $0 < \epsilon_j \leq \pi/2$ (see the remark below),

$$\kappa_j = \cos(\frac{\pi}{2} - \epsilon_j) > 0, \quad \text{for } 1 \leq j \leq r. \tag{2}$$

Remark: In order to see that $0 < \epsilon_j \leq \pi/2$, one may verify that if $r \geq 3$,

$$\begin{aligned} \epsilon_j &= \theta_1 \text{ if } j = 1 \\ &= \theta_j - \theta_{j-1} \text{ if } j = 2, 3, \cdots r-1 \\ &= \frac{\pi}{2} - \theta_{r-1} \text{ if } j = r. \end{aligned}$$

Of course if $r = 1, 2$ then it is very easy to verify that $0 < \epsilon_j \leq \pi/2$.

2 Proof of Theorem 1.1

We note that $\mathcal{M}f(x) \leq \mathcal{M}|f|(x)$ and so while proving the weak-type estimate for $\mathcal{M}$, we may assume that $f \geq 0$. As a result we may further assume that

$$\mathcal{M}f(x) = \sup_{p,q \geq K_0} 2^{p+q} \left| \int_{2^{-p}}^{2^{-p+1}} \int_{2^{-q}}^{2^{-q+1}} f(x - \phi(s,t)) \mathrm{d}s\mathrm{d}t \right|,$$

where K_0 is a sufficiently large positive integer (depending upon c in the definition of $\mathcal{M}$).
As in the case of polynomial, here also, we first discuss the proof of Theorem 1.1 assuming that v_1 and/or v_r do not have a vanishing x-coordinate and/or y-coordinate respectively and later discuss the proof of these special cases.
The main essence of the proof discussed in [5] was the decomposition of $\mathcal{M}$ and then the proofs of the Lemma 4.2, 4.3, 4.4 (Lemma 6.3, 6.4 and 6.5 in the special cases). We ask the reader to verify that if we prove these lemmas for the real-analytic case, then the rest of the proof of Theorem 1.1 follows as it is in [5]. Thus we only need to prove these lemmas for the real-analytic case. But Lemma 4.3 and 4.4 are simply the consequences of Lemma 4.2. Hence our only job here, is to prove Lemma 4.2 for the real-analytic case. Going through the proof of this lemma in [5], we notice that its proof works for the real-analytic case if we establish the following:

(I) Some partial derivative (of order greater than 1) of

$$\tilde{\phi}_j(s,t) = a_{m_j,n_j} s^{m_j} t^{n_j} + \sum_{v=(m,n)\in\Lambda\backslash\{v_j\}} 2^{-(p,q)\cdot(v-v_j)} a_{m,n} s^m t^n$$

is uniformly (in terms of (p,q)) bounded below for large p,q and;
(II) The number

$$\beta_j = \frac{1}{d} \inf_{v\in\Lambda\backslash\{v_j\}} \{(\overline{n}_{j-1}+\overline{n}_j)\cdot(v-v_j)\} > 0.$$

Note that $\beta_j > 0$ follows because geometrically it is easy to see that for each $v \in \Lambda\backslash\{v_j\}$

$$(\overline{n}_{j-1}+\overline{n}_j)\cdot(v-v_j) \geq \kappa_j.$$

For (I) we argue as follows.
Corresponding to each vertex $v_j = (m_j, n_j)$, define the sets

$$\Lambda(j) = \{v = (m,n) \in \Lambda : m \geq m_j \text{ and } n \geq n_j\} \text{ and } \Lambda^\star(j) = \Lambda(j)\backslash\{v_j\}.$$

Then the v_jth derivative of $\tilde{\phi}_j(s,t)$ can be written as

$$\partial^{v_j}\tilde{\phi}_j(s,t) = c_{v_j} a_{m_j,n_j} + \sum_{v=(m,n)\in\Lambda^\star(j)} 2^{-(p,q)\cdot(v-v_j)} c_v a_{m,n} s^{m-m_j} t^{n-n_j},$$

where c_v are positive integers (which can be computed easily). Thus in order to show that $\partial^{v_j}\tilde{\phi}_j(s,t)$ is bounded below, it is enough to show that $(p,q)\cdot(v-v_j)$ is large (for large p,q). Now we see that if $2 \leq j \leq r$, then for any $v \in \Lambda^\star(j)$,

$$\overline{n}_j\cdot(v-v_j) \geq \rho_j \text{ and } \overline{n}_{j-1}\cdot(v-v_j) \geq \rho_{j-1}.$$

So if $2 \leq j \leq r$, then for $(p,q) \in Z_1^N(j)$ and $v \in \Lambda^\star(j)$, we have

$$\begin{aligned}(p,q)\cdot(v-v_j) &= \frac{k}{d}\overline{n}_{j-1}\cdot(v-v_j) + \frac{N}{d}(\overline{n}_{j-1}+\overline{n}_j)\cdot(v-v_j)\\ &\geq (\frac{k+N}{d})\overline{n}_{j-1}\cdot(v-v_j)\\ &\geq \frac{\rho_{j-1}}{d}(k+N).\end{aligned}$$

Now note that if p and q are sufficiently large (recall that we are only considering $p,q \geq K_0$) then $k+N$ is large. So we may now assert that if $2 \leq j \leq r$, $(p,q) \in Z_1^N(j)$ and if p and q are sufficiently large (i.e. $k+N$ is large) then $\partial^{v_j}\tilde{\phi}_j(s,t)$ is uniformly (in terms of (p,q)) bounded below. When $j=1$ we argue as follows.
For $(p,q) \in Z_1^N(1)$ and $v \in \Lambda^\star(1)$,

$$\begin{aligned}(p,q)\cdot(v-v_1) &= \frac{k}{d}\overline{n}_0\cdot(v-v_1)+\frac{N}{d}(\overline{n}_0+\overline{n}_1)\cdot(v-v_1)\\ &\geq \frac{N}{d}(\overline{n}_0+\overline{n}_1)\cdot(v-v_1)\\ &\geq \frac{\kappa_1}{d}(N). \end{aligned}\tag{3}$$

Now since $(p,q) \in Z_1^N(1)$, we have

$$(p,q)=\frac{k}{d}\overline{n}_0+\frac{N}{d}(\overline{n}_0+\overline{n}_1).\tag{4}$$

But $\overline{n}_0=(1,0)$ and so it follows from (4) that q is proportional to N. Thus if q is large then we may assume that N is large. In view of (3), we can now assert that $(p,q)\cdot(v-v_1)$ is large for large q, and hence $\partial^{v_1}\tilde{\phi}_1(s,t)$ is bounded below.
The proof of Lemma 4.2 for the polynomial case can now be easily adapted to the real-analytic case. Now let us discuss the proof of the special cases. We only discuss the case where v_1 has a vanishing x-coordinate. The proof of the case where v_r has a vanishing y-coordinate is similar.
Note that due to our hypothesis, the set $\{v=(m,n)\in\Lambda : m=0\}$ is simply the singleton set $\{v_1=(0,n_1)\}$ now. So (following the notations of Section 6 in [5]) $\tilde{P}_0(t)$ is simply the monomial $a_{0,n_1}t^{n_1}$. This makes our job slightly easier here as compared to the polynomial case. (In the polynomial case we did not have such hypothesis and so the proof of special cases becomes more technical there).
Going through the Section 6 of [5] , it is once again clear that the proof of this special case follows if we establish Lemma 6.3, 6.4 and 6.5 for the real-analytic case. But Lemma 6.4 and 6.5 are simply the consequences of Lemma 6.3. Hence we only need to establish Lemma 6.3 for the real-analytic case. The proof of Lemma 6.3 for the polynomial case can be modified to make it work for the real-analytic case. But this would be a longer route and unnecessarily more technical. We rather follow the techniques of Lemma 4.2 again.
Following the notations of Section 6 in [5] (and writing $\tilde{\phi}(s,t)$ instead of $\tilde{P}(s,t)$) we have

$$\begin{aligned}< f,\nu^{k,(l)} > &= \int\!\!\int [f(\tilde{\phi}(s,t))-f(a_{0,n_1}t^{n_1})]\eta(s)\eta(t)\mathrm{d}s\mathrm{d}t\\ &= \int\!\!\int f(\tilde{\phi}(s,t))\eta(s)\eta(t)\mathrm{d}s\mathrm{d}t-\int\!\!\int f(a_{0,n_1}t^{n_1})\eta(s)\eta(t)\mathrm{d}s\mathrm{d}t\\ &:= < f,\nu_1^{k,(l)} > - < f,\nu_2^{k,(l)} > \end{aligned}$$

for those l's for which $\frac{k}{d}\overline{n}_0+\frac{l}{d}\overline{n}_1=(p,q)\in Z^k(1)$, and where

$$\tilde{\phi}(s,t)=a_{0,n_1}t^{n_1}+\sum_{v=(m,n)\in\Lambda\setminus\{v_1\}}2^{-(p,q)\cdot(v-v_1)}a_{m,n}s^m t^n.$$

Thus

$$\partial^{v_1}\tilde{\phi}(s,t) = c_{v_1}a_{0,n_1} + \sum_{v=(m,n)\in\Lambda^\star(1)} 2^{-(p,q)\cdot(v-v_1)}c_v a_{m,n}s^m t^{n-n_1}.$$

We now have to show that for $(p,q) \in Z^k(1)$, $\partial^{v_1}\tilde{\phi}(s,t)$ is bounded below (for large p,q). Once again, this will be done by showing that $(p,q)\cdot(v-v_1)$ is large for $v \in \Lambda^\star(1)$. For this we need the following remark.

Remark: Since

$$(p,q) = \frac{k}{d}\overline{n}_0 + \frac{l}{d}\overline{n}_1 \text{ and } \overline{n}_0 = (1,0),$$

l is proportional to q. Thus we may assume that l is large if q is large. Additionally, in the case $r = 1$, since $\overline{n}_1 = (0,1)$, we also have k proportional to p. So when $r = 1$, we may assume that k is large if p is large.

Now if $r \geq 2$, we see that for any $v \in \Lambda^\star(1)$,

$$\overline{n}_1 \cdot (v - v_1) \geq \rho_1.$$

So if $r \geq 2$, then for $(p,q) \in Z^k(1)$ and $v \in \Lambda^\star(1)$,

$$(p,q)\cdot(v-v_1) = \frac{k}{d}\overline{n}_0\cdot(v-v_1) + \frac{l}{d}\overline{n}_1\cdot(v-v_1) \geq \frac{\rho_1}{d}(l). \tag{5}$$

In the case $r = 1$, we argue as follows.

Since $\overline{n}_0 = (1,0)$ and $v_1 = (0,n_1)$, for any $v \in \Lambda\backslash\{v_1\}$, $\overline{n}_0\cdot(v-v_1)$ is actually the first component of v. Further, because v is not on Y-axis (due to our hypothesis that Λ has at the most one point on Y-axis and it is v_1 here), the first component of v is at least 1. So in this case for $(p,q) \in Z^k(1)$ and $v \in \Lambda^\star(1)$, we have

$$(p,q)\cdot(v-v_1) \geq \frac{1}{d}(k). \tag{6}$$

From (5), (6) and the remark above, it follows that when p,q are large, $\partial^{v_1}\tilde{\phi}(s,t)$ is bounded below (within the support of $\eta(s)\eta(t)$). This facilitates in applying Proposition 7.2 of [4] which helps us to assert that $\nu^{k,(l)}$ is absolutley continuous and satisfy

$$\int |\nu^{k,(l)}(x-y) - \nu^{k,(l)}(x)|\mathrm{d}x \leq A|y|^\epsilon, \tag{7}$$

for all $y \in R$, ϵ sufficiently small and A independent of k and l. However in order to conclude the proof of Lemma 6.3 (for the real-analytic case) we need estimate (7) with some decay in k. It is enough to get this decay for large k. This is achieved as in Lemma 4.2 once we show that $\nabla\tilde{\phi}(s,t)$ is bounded below (within the support of $\eta(s)\eta(t)$). For this it is enough to see that there exists a positive number say β, so that if $(p,q) \in Z^k(1)$ and $v \in \Lambda\backslash\ \{v_1\}$, then

$$(p,q)\cdot(v-v_1) \geq \frac{k}{d}\overline{n}_0\cdot(v-v_1) \geq \beta k.$$

But as explained above, $\overline{n}_0\cdot(v-v_1) \geq 1$ for all $v \in \Lambda\backslash\ \{v_1\}$, so that

$$\beta = \frac{1}{d}$$

works here and this enables to complete the proof of Theorem 1.1 in the special case. Note that, without the assumption that Λ has at the most one point on Y-axis, the whole argument (in the special case) breaks down because $\overline{n}_0\cdot(v-v_1)$ could be 0 then for some $v \in \Lambda\backslash\ \{v_1\}$. Similarly, in the proof of special case where v_r is on the X-axis, we require the assumption that Λ has at the most one point on X-axis.

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