Mathematics Today Vol.30 (June & December 2014; Published in June 2015) 91-98
ISSN 0976-3228

A note on topological divisors of zero

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(Acceptance Date: May 14, 2015)

Abstract

We introduce a new concept of topological divisor of zero in locally m-convex algebras. For this concept, we show that conditions which in a Banach algebra are sufficient for an element to be a topological divisor of zero are also sufficient in complete locally m-convex algebras. We also give an answer to a problem concerning the equivalence of three notions of topological divisor of zero in the negative by providing a wide range of counterexamples.

Keywords: m-convex algebra, Fréchet algebra, Arens-Michael representation, (strong) topological divisor of zero

Mathematics Subject Classification (2000): Primary 46J05; Secondary 13F25

All algebras in this note are commutative, complex, locally multiplicative convex algebras (shortly: m-convex algebras) with identity unless otherwise specified. If A is such an algebra, then its topology is defined by means of a family (p_k) of submultiplicative seminorms (which can be taken to be saturated without loss of generality). The basic theory of m-convex algebras can be found in [8]. The principal tool for studying such algebras is the Arens-Michael representation, in which a complete m-convex algebra A is given by an inverse limit of Banach algebras A_k (see [9, §2]; given for Fréchet algebras). It should be noted that what we write as A_k appears as $\overline{A_k}$ in our main reference [8].

The first aim of the present note is to discuss a problem of Michael: whether the two concepts — strong topological divisor of zero (shortly: s.t.d.z.) due to Arens and topological divisor of zero (shortly: t.d.z.) due to Michael, defined below, — are equivalent [8, pp. 46-7]. We note that Kuczma in [6] already showed that $\mathbb{C}[[X]]$ has no s.t.d.z.. However, to show the non-equivalence of two concepts, one just needs to show the existence of an element which is t.d.z. but not s.t.d.z.. Thus, it is of interest to investigate that which algebras possess such an element. Surprisingly, a wide range of unital commutative, complete m-convex algebras possess such an element. (See Theorems 1-3 below.)

A *Fréchet algebra of power series* is a subalgebra A of $\mathbb{C}[[X]]$ such that A is a Fréchet algebra containing the indeterminate X [5]. Recently, Fréchet algebra of power series—and more generally, the power series ideas in general Fréchet algebras—have acquired significance in understanding the structure of Fréchet algebras ([1], [3], [4], [5], [9], [10]). The ideal structure in Fréchet algebras has been extensively studied in [4], [9], [10], [12], [13].

A unital topological algebra A is called a Q-algebra provided the set A^{-1} of invertible elements is open. Let A be a topological algebra and let $x \in A$. Let L_x and R_x denote the linear operators $A \to A$ of left- and right-multiplication by x. A non-zero element $x \in A$ is called a *strong topological divisor of zero* (shortly: s.t.d.z.) if either R_x or L_x is not an isomorphism into, i.e. a linear homeomorphism of A onto Ax or xA. If A is metrisable, an alternative description of s.t.d.z.'s is: an element $x \in A$ is a s.t.d.z. if and only if $x \neq 0$ and there exists a sequence (x_n) in A with x_n does not converge to 0 and either $xx_n \to 0$ or $x_n x \to 0$. A non-zero element x of an m-convex algebra A will be called a *topological divisor of zero* (shortly: t.d.z.) if, whenever (p_k) is a defining family of seminorms for the topology of A, there exists an i such that x_i is a s.t.d.z. in A_i. It follows that an element which is a s.t.d.z. cannot be invertible since s.t.d.z. implies t.d.z..

The following facts are well-known in the theory of complete m-convex algebras: Let A be a complete m-convex algebra with identity e, and let $x \in A$ such that $x \neq 0$. Then

(1) if x is in the closure of the set of all invertible elements but is not itself invertible (the latter assumption can be omitted if A is a Q-algebra, in particular, if A is Banach), then x is a t.d.z. [8, Proposition 11.6];

(2) if $\lambda \in \mathrm{Sp}_A(x)$, where $\mathrm{Sp}_A(x)$ denotes the spectrum of x, is a boundary point of

$\mathrm{Sp}_A(x)$, then $x - \lambda e$ is a t.d.z. [8, Proposition 11.8]. Consequently, the radical of A consists entirely of t.d.z.'s and 0;

(3) if A is a Q-algebra containing no non-zero t.d.z., then A is isomorphic to $\mathbb{C}$ [8, Proposition 11.12].

Moreover, if A is a Banach algebra, then the concepts of t.d.z. and s.t.d.z. agree [8, Proposition 11.3]. However, the following theorem shows that these two concepts are not equivalent in a complete non-Banach m-convex algebras.

THEOREM 1. *Let $(A, (p_k))$ be a unital, commutative, semisimple, complete metrisable m-convex algebra with the Arens-Michael isomorphism $A \cong \varprojlim(A_k; d_k)$, and which has a non-isolated principal maximal ideal generated by $x \in A$. Suppose $x_i \in bdry(A_i^{-1})$ for some i. Then x is a t.d.z. but is not a s.t.d.z..*

REMARKS. 1. Observe that the analogue of this result is also true for a non-unital algebra. This follows from the following, rather obvious, fact: a commutative topological algebra has all ideals closed if and only if its unitization has all ideals closed. Also if M is a proper principal ideal in A, then it is a proper principal ideal in the unitization of A after the natural imbedding of A in this unitization. In the non-unital case, we consider the set of all quasi-invertible elements in place of A_i^{-1}. We do not know, however, whether the non-commutative version of Theorem 1 holds true. Also we do not know whether there exists a non-unital, commutative complete m-convex algebra (even a Fréchet algebra) with all its principal maximal ideals dense in the algebra. This question reminds us of the still unsolved, old and famous problem: whether every infinite-dimensional Banach algebra has a proper closed ideal. Recently, Read constructed a non-unital, commutative, incomplete normed algebra in which all proper subalgebras are dense [11].

2. The hypotheses of the theorem are not only simple and aesthetically pleasing; but also hold good in several important algebras of analysis (see Examples.)

PROOF OF THEOREM 1: Clearly, Ax is closed being a principal maximal ideal, by [8, Theorem 5.4]. Let ϕ be the point of $M(A)$ corresponding to Ax. Evidently, R_x is a continuous linear mapping of A onto Ax. Further, suppose that $yx = 0$ for some $y \in A$. Since $\mathrm{hull}(\hat{x}) = \{\phi\}$, $\hat{y}$ must vanish off $\{\phi\}$. Since $\{\phi\}$ is not isolated and $\hat{y}$ is continuous, $\hat{y}$ vanishes identically on $M(A)$. Therefore, R_x is a one-to-one mapping. Now, by the open mapping theorem, R_x is a homeomorphism of A onto Ax, and so x is not a s.t.d.z.. Rest is an easy exercise for the reader. □

A little reflection also reveals the following result since R_x is automatically one to one mapping in this case:

THEOREM 2. *Let* $(A, (p_k))$ *be a unital, commutative, complete metrisable* m*-convex algebra with the Arens-Michael isomorphism* $A \cong \varprojlim(A_k;\, d_k)$*, and which is an integral domain possessing a principal maximal ideal generated by* $x \in A$*. Suppose* $x_i \in bdry(A_i^{-1})$ *for some* i*. Then* x *is a t.d.z. but is not a s.t.d.z..*

REMARK. It is clear that Theorems 1 and 2 are independent of the Arens-Michael representation chosen, in the sense that if (q_k) is any other sequence of seminorms defining the same Fréchet algebra topology of A, then the theorems are valid with that Arens-Michael representation. Of course, the index i, obtained using the sequence (q_k), may be different. This is possible because the sequences (p_k) and (q_k) are equivalent, generating the same Fréchet algebra topology of A.

Using elementary results of local algebra, we have the following special case which generalizes the theorem of Kuczma [6].

THEOREM 3. *Let* A *be a local, complete* m*-convex algebra which is an integral domain. Then* A *has no s.t.d.z..*

PROOF: Clearly, A is a commutative algebra with identity e. Let M be the unique maximal ideal of A. Then $M = \mathrm{rad}A = A \setminus A^{-1}$, and A^{-1} is dense in A since if $y \in M$, then $y_n = y + \frac{1}{n}e \in A^{-1}$ and $y_n \to y$. Now, by [8, Proposition 11.6], $y\,(\neq 0) \in M$ is a t.d.z. in A; but it is not a s.t.d.z.. Certainly, no element of A^{-1} is (s.)t.d.z.. □

As a consequence of the theorem above, we have the following

COROLLARY 4. *Every local Fréchet algebra of power series has no s.t.d.z..*

EXAMPLES. The initial range of examples are Fréchet algebras of power series A such that X is a power series generator for A (see [3] or [9]). By [3, Theorem 2.1], A is either $\mathbb{C}[[X]]$ or the Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, W)$ for an increasing sequence W of weights on $\mathbb{Z}^+$. Further, if A $(\neq \mathbb{C}[[X]])$ is a local algebra; i.e. W is a sequence of local weights on $\mathbb{Z}^+$, then, by Corollary 4, A has no s.t.d.z.. If A is a semisimple, non Q-algebra, then A is isomorphic to either $\mathrm{Hol}(U)$ or the algebra of entire functions, and if A is a Q-algebra and satisfies certain condition (*), then A is isomorphic to $A^\infty(\Gamma)$, by [3, Theorem 3.7]. In all the three cases, A satisfies hypotheses of Theorem 1, showing that X is not a s.t.d.z.. Like $A^\infty(\Gamma)$, $F(W)$ are the other examples (see [4, §4] for more details).

We remark that the completeness assumption is essential in Theorems 1 and 2 since $F(W)$ equipped with the supremum norm $|\cdot|_\infty$ is an incomplete normed Q-algebra (see [13, Example 8] for more details), and it is well-known that, for normed algebras, both concepts of t.d.z. are equivalent. Similarly, $\mathbb{C}[[X]]$ is a normable (but not Banachable) algebra (see [1, Theorem 2]), and so we cannot remove the completeness assumption in Theorem 3.

Thus, Michael introduces in [8] a weaker concept to show that conditions which in a Banach algebra are sufficient for an element to be a t.d.z. are also sufficient in complete m-convex algebras, but the concept has a high degree of non-equivalency with the concept of s.t.d.z.. In such a situation, it is of interest to ask whether one can replace a weaker concept of Michael by another concept of t.d.z. so that one can investigate: (1) whether a new concept is equivalent to the either one, and (2) whether one can extend the relevant theorems for Banach algebras to complete m-convex algebras (see [8, Propositions 11.6 - 11.12]). We note that the notion of topological divisor of zero is one of the important notions in the theory of Banach algebras.

Here is a new concept of t.d.z. in an m-convex algebra A: a non-zero element $x \in A$ is called a *topological divisor of zero* in author's sense (shortly: P-t.d.z.) if, whenever (p_k) is a defining family of seminorms for the topology of A, there exists an i such that x_m is a t.d.z. in A_m for all $m \geq i$. Clearly, an element which is a P-t.d.z. is also a t.d.z., and so it cannot be invertible. Thus this new concept (which is less aesthetic, and which is meaningful only for locally m-convex algebras) is stronger than the concept of Michael. As a consequence, we do not only have freedom of following the proofs of propositions of [8, §11], but also they do go through (with a minor change in some cases) for the smaller class of elements which are P-t.d.z. in A.

Next, we introduce a weaker concept of (quasi) regular elements: a non-zero element $x \in A$ is called *weakly (quasi) regular* in A if, for each family (p_k) of seminorms defining the topology of A, there exists an i such that x_i is (quasi) regular in A_i. Clearly, an element which is (quasi) regular is also weakly (quasi) regular, and both concepts agree for normed algebras. The following four propositions provide sufficient conditions for the existence of an element which is a P-t.d.z.; in all the four propositions, x is in A a complete, m-convex algebra.

PROPOSITION 5. *If x is in the closure of the set of quasi regular elements of A but is not itself weakly quasi regular, then $e+x$ is a P-t.d.z..*

PROPOSITION 6. *If $-\lambda x$ is quasi regular for arbitrarily large λ, then either x is a P-t.d.z. or x is weakly regular.*

PROPOSITION 7. *If $\lambda \in Sp_A(x)\,(= \bigcup_i Sp_{A_i}(x_i))$ is a boundary point of $Sp_A(x)$, then $x-\lambda e$ is a P-t.d.z..*

We note that $x-\lambda e$, which is a t.d.z., by [8, Proposition 11.8], is actually a P-t.d.z. in Proposition 7. Hence we have the following

PROPOSITION 8. *If $x-\lambda e$ is a s.t.d.z., and $\lambda \in$ bdry($Sp_A(x)$), then $x-\lambda e$ is a P-t.d.z.. The converse holds if A is a Banach algebra.*

We note that Proposition 11.5 of [8] does go through for this new concept of P-t.d.z., and that every m-convex algebra can be completed in a given topology. So Proposition 8, primarily addressed for complete m-convex algebras, also holds true for m-convex algebras.

Next, we investigate just how the three concepts relate to each other. We first remark that all the three concepts of "topological divisor of zero" agree for normed algebra. However, in the more general case of m-convex algebras, the concept of P-t.d.z. is not equivalent to that of (s.)t.d.z. as witnessed in the case of the concepts of t.d.z. and s.t.d.z.. In fact, Theorems 1 and 2 hold true in this case provided that the hypothesis "$x_i \in \mathrm{bdry}(A_i^{-1})$ for some i" is replaced by the hypothesis "$x_i \in \mathrm{bdry}(A_i^{-1})$ for all i" in these theorems; also, Theorem 3 holds true by noticing that $y \in M$ in the proof of Theorem 3 is, indeed, a P-t.d.z. in A. For example, take $A = F(W)$ (in particular, $A^\infty(\Gamma)$) as considered in Examples above. Let $\lambda \in \Gamma$. Since Γ is the boundary of the maximal ideal space of A_m for each m, $X-\lambda e$ is a P-t.d.z. in A [10, 4.2]. In the local case, take $A = \ell^1(\mathbb{Z}^+, W)$, $W = (\omega_n)$ is an increasing sequence of local weights on $\mathbb{Z}^+$, as in Example 1.2 of [3]. Then X is, indeed, a P-t.d.z. in A (see [10, 4.2]).

The non-equivalence of P-t.d.z. with t.d.z. is quite clear — take $A = \mathrm{Hol}(U)$, with a sequence (p_n) of sup-norms defined by $p_n(f) = \sup_{|z|\le 1-\frac{1}{n}} \mid f(z) \mid$, then $f(z) = z-(1-\frac{1}{m})$ is a s.t.d.z. in A_m but not in A_{m-1}.

Thus, summarizing the inter-relations among these three concepts of t.d.z., we have s.t.d.z. implies t.d.z., s.t.d.z. implies P-t.d.z. conditionally, and P-t.d.z. implies t.d.z.;

but the converse does not hold in all the three cases. Next, from Proposition 7 and the analogue of [8, Proposition 11.5], we conclude the following

PROPOSITION 9. *Let $\hat{A}$ be a locally m-convex algebra with complete subalgebra A, and let $x \in A$. If $\lambda \in Sp_A(x)$ is a boundary point of $Sp_A(x)$, then $\lambda \in Sp_{\hat{A}}(x)$.*

Note that every element of a Q-algebra has a compact spectrum [8].

PROPOSITION 10. *Let A be a locally m-convex algebra whose completion $\overline{A}$ is a Q-algebra, and let $x \in A$. Then $x - \lambda e$ is a P-t.d.z. for some λ.*

From the two examples above [10, 4.2] it is natural to suspect that the existence of an element which is a P-t.d.z. occurs only in the class of complete, m-convex Q-algebra; but the converse of Proposition 10 does not hold. For example, take $A = H(\Gamma[r_2, r_1))$ (respectively, $H(\Gamma(r_2, r_1]))$ as in §1 of [2]. Then A is a Fréchet algebra having a Laurent series generator z in the sense of [2], with $M(A) = \Gamma[r_2, r_1))$ (respectively, $\Gamma(r_2, r_1]$), the annuli defined with appropriate r_1 and r_2. Let $\lambda = r_2 e^{i\theta}$ (respectively, $r_1 e^{i\theta}$), where $0 \leq \theta \leq 2\pi$. Then $z - \lambda$ is a P-t.d.z., but A is not a Q-algebra as $M(A)$ is not compact [12].

Below we admit that the argument of Proposition 11.12 of [8] still applies using Proposition 10 above.

PROPOSITION 11. *Let A be as in Proposition 10 containing no non-zero P-t.d.z.. Then A is isomorphic to $\mathbb{C}$.*

Finally, all the three concepts of t.d.z. are topological ones, and so, would follow for every choice of the defining family (p_k). However, one may have two nonequivalent topologies on the same algebra A, and it may be possible to find an element $x \in A$ such that x is a t.d.z. under one topology, and is not a t.d.z. under the other topology on A. In the case of Banach algebras, Loy in [7] had constructed commutative Banach algebras with non-unique complete norm topologies. However, we do not know whether there exists an element in these algebras such that it is a t.d.z. under one complete norm, and is not a t.d.z. under the other. Thus, it would be interesting to find an example of a Banach (or even Fréchet) algebra with two nonequivalent complete norm (or even Fréchet algebra) topologies, and which possesses such an element.

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