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Some results on a spanning subgraph of the intersection graph of ideals of a commutative ring

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Abstract

The rings considered in this article are commutative with identity and which admit at least one nonzero proper ideal. For a ring R, we denote by $I(R)$, the set of all proper ideals of R and let $I(R)^* = I(R)\backslash\{(0)\}$. In this article, for any ring R, we associate an undirected simple graph, denoted by $H(R)$, whose vertex set is $I(R)^*$ and distinct vertices I, J are joined by an edge in this graph if and only if $IJ \neq (0)$. For a ring R, we determine necessary and sufficient conditions in order that $H(R)$ is connected and also find its diameter when it is connected. We prove that $girth(H(R))$ is either equal to 3 or ∞. Moreover, we classify the rings R for which $girth(H(R)) = 3$. Furthermore, we determine necessary and sufficient conditions in order that $H(R)$ is complemented.

Keywords: B-prime of (0), diameter, girth, complemented graph.

Mathematics Subject Classification 2010: 13A15

1 Introduction

The rings considered in this article are commutative with identity $1 \neq 0$. The idea of associating a graph with a ring R and studying the interplay between the ring theoretic properties of R and the graph theoretic properties of a graph associated with it was initiated by I. Beck in [10]. Subsequently, a lot of research activity has been carried out by several researchers in this area (see, for example, [3, 4, 5, 6, 8, 17, 18, 20]). The study of intersection graph of ideals of a ring has begun with the work of Chakrabarthy, Ghosh, Mukherjee, and Sen [13]. Let R be a ring with identity which is not necessarily commutative and which admits at least one nonzero proper left ideal. Recall from [13] that the intersection graph of ideals of R, denoted by $G(R)$, is an undirected simple graph whose vertex set is the set of all nonzero proper left ideals of R and two distinct vertices I, J are joined by an edge in this graph if and only if $I \cap J \neq (0)$. The intersection graph of ideals of a ring was studied by seveal researchers (see, for example [1, 15, 19]). Let R be a commutative ring with identity which is not a field. Let $G(R)$ denote the intersection graph of ideals of R. Motivated by the works presented in [1, 13, 15, 19] on $G(R)$, in this article, we associate an undirected simple graph with R, denoted by $H(R)$, whose vertex set is the set of all nonzero proper ideals of R and two distinct vertices I, J are joined by an edge in $H(R)$ if and only if $IJ \neq (0)$. Note that for any ideals I, J of a ring R, $IJ \subseteq I \cap J$. Thus $IJ \neq (0)$ implies that $I \cap J \neq (0)$. Moreover, observe that the vertex set of $G(R) =$ the vertex set of $H(R)$. Hence, it is clear that $H(R)$ is a spanning subgraph of $G(R)$. The purpose of this article is to study the influence of some graph theoretic parameters of $H(R)$ on the ring structure of R and vice-versa.

The graphs considered in this article are undirected. Let $G = (V, E)$ be a graph. Let $a, b \in V$, $a \neq b$. Recall that the distance between a and b, denoted by $d(a, b)$, is defined as the length of a shortest path between a and b in G if such a path exists; otherwise, $d(a, b) = \infty$. We define $d(a, a) = 0$. G is said to be connected, if for any distinct $a, b \in V$, there exists a path in G between a and b. The diameter of a connected graph $G = (V, E)$, denoted by $diam(G)$, is defined as $diam(G) = sup\{d(a, b)|a, b \in V\}$.

Let $G = (V, E)$ be a graph such that G contains a cycle. Recall from [9, p.159] that the girth of G, denoted by $girth(G)$, is equal to the length of a shortest cycle in G. If a graph G does not contain any cycle, then we define $girth(G) = \infty$.

Let $G = (V, E)$ be a graph. Recall from [5, 17] that two distinct vertices u, v of G are said to be orthogonal, written $u \perp v$, if u and v are adjacent in G and there is no vertex w of G which is adjacent to both u and v in G; that is, the edge $u - v$ is not the edge of any triangle in G. A vertex v of G is said to be a complement of u if $u \perp v$ [5]. Moreover, recall from [5] that G is complemented if each vertex of G admits a complement in G. Furthermore, G is said to be uniquely complemented, if G is complemented and whenever the vertices u, v, w of G are such that $u \perp v$ and $u \perp w$, then a vertex x of G is adjacent to v in G if and only if x is adjacent to w in G. Let $\Gamma(R)$ denote the zero-divisor graph of a ring R. In Section 3 of [5], D.F. Anderson, R. Levy and J. Shapiro determined rings R for which the zero-divisor graphs are complemented or uniquely complemented. In this article, we characterize rings R such that $H(R)$ is complemented.

Let I be an ideal of a ring R. Recall from [14] that a prime ideal P of R is said to be an associated prime of I in the sense of Bourbaki, if $P = (I :_R x)$ for some $x \in R$. In this case, we say that P is a B-prime of I in R. A ring R is said to be quasilocal if R has a unique maximal ideal. A Noetherian quasilocal ring is referred to as a local ring.

Let R be a ring. Recall that an element $a \in R$ is said to be nilpotent if $a^n = 0$ for some $n \geq 1$. It is well-known that the set of all nilpotent elements of R forms an ideal and is called the nilradical of R. We denote the nilradical of R by $nil(R)$. R is said to be reduced if $nil(R) = (0)$. Observe that for ideals I, J of a reduced ring R, $I \cap J \neq (0)$ if and only if $IJ \neq (0)$. Hence for any reduced ring R, $H(R) = G(R)$. Whenever a set A is a subset of a set B and $A \neq B$, we denote it symbolically using the notation $A \subset B$. For a set A, we denote the cardinality of A using the notation $|A|$. We denote the set of all zero-divisors of a ring R by $Z(R)$.

In Section 2, we prove some properties of $H(R)$, where R is a ring which admits a maximal ideal M such that M is not a B-prime of (0) in R. In Section 3, we consider rings R such that R is quasilocal. Let M denote the unique maximal ideal of R. With the assumption that $H(R)$ contains at least two vertices, we determine in Proposition 3.1 necessary and sufficient conditions in order that $H(R)$ is connected. If R is not an integral domain and if $H(R)$ is connected, then it is verified in Remark 3.2 that $diam(H(R)) = 2$. If the unique maximal ideal M of R is not a B-prime of (0) in R, then it is observed in Proposition 3.3 that no vertex of $H(R)$ admits a complement in $H(R)$ and moreover, $girth(H(R)) = 3$. In Proposition 3.4, we describe the properties of $H(R)$ under the assumption that the unique maximal ideal M of R is a B-prime of (0) in R.

In Section 4, we consider rings R such that R has exactly two maximal ideals. Let $\{M_1, M_2\}$ denote the set of all maximal ideals of R. In Proposition 4.4, we determine necessary and sufficient conditions in order that $H(R)$ is connected. Under the assumption that $H(R)$ is connected, Proposition 4.5 describes $diam(H(R))$. In Theorem 4.12, we classify rings R such that $H(R)$ is complemented. Moreover, in Theorem 4.13, we determine necessary and sufficient conditions in order that $H(R)$ contains a cycle.

In Section 5, we consider rings R such that R has more than two maximal ideals and investigate the properties of $H(R)$. In Section 6 of this article, we give a brief description of the results that are proved in Sections 3, 4, and 5 of this article.

2 Some basic results

In this section, we state and prove some basic results that are needed for proving some results of this article. We make use of some ideas that are used in the proofs of results from [21].

Lemma 2.1. Let M be a maximal ideal of a ring R such that M is not a B-prime of (0) in R. Then $M \not\subseteq ((0) :_R I)$ for any nonzero ideal I of R.

Proof. Suppose that $M \subseteq ((0) :_R I)$ for some nonzero ideal I of R. Let $a \in I\backslash\{0\}$. Note that $M \subseteq ((0) :_R a)$. As $((0) :_R a) \neq R$ and M is a maximal ideal of R, it follows that $M = ((0) :_R a)$. This is in contradiction to the assumption that M is not a B-prime of (0) in R. Therefore, we obtain the desired conclusion. □

Lemma 2.2. Let M be a maximal ideal of a ring R such that M is not a B-prime of (0) in R. Then $H(R)$ is connected and moreover, $diam(H(R)) \leq 2$.

Proof. Let I, J be distinct vertices of $H(R)$. If $IJ \neq (0)$, then there is an edge of $H(R)$ joining I and J. Suppose that $IJ = (0)$. Since M is not a B-prime of (0) in R, it follows from Lemma 2.1 that $IM \neq (0)$ and $JM \neq (0)$. Hence $I - M - J$ is a path of length 2 in $H(R)$ between I and J. This proves that $H(R)$ is connected and $diam(H(R)) \leq 2$. □

Lemma 2.3. Let M be a maximal ideal of a ring R such that M is not a B-prime of (0) in R. Then any edge of $H(R)$ is an edge of a triangle in $H(R)$.

Proof. Let $I - J$ be any edge of $H(R)$. Since M is not a B-prime of (0) in R and $IJ \neq (0)$, we obtain from Lemma 2.1 that $IJM \neq (0)$. Hence $IM \neq (0)$ and $JM \neq (0)$. We consider the following cases:

Case(i) $M \notin \{I, J\}$

In this case, it is clear that $I - J - M - I$ is a cycle of length 3 in $H(R)$.

Case(ii) $M \in \{I, J\}$

Without loss of generality, we may assume that $M = I$. Since $JM \neq (0)$ and $M \not\subseteq J$, we obtain that $M \not\subseteq ((0) :_R JM) \cup J$. Hence there exists $m \in M\backslash J$ such that $MJm \neq (0)$. If $M \neq Rm$, then $M - J - Rm - M$ is a cycle of length 3 in $H(R)$. Suppose that $M = Rm$. We assert that $m \notin Z(R)$. For if $m \in Z(R)$, then there exists $y \in R\backslash\{0\}$ such that $my = 0$. This implies that $M = ((0) :_R y)$ is a B-prime of (0) in R. This is a contradiction. Therefore, $m \notin Z(R)$. As m is not a unit in R and $m \notin Z(R)$, it follows that $Rm^i \neq Rm^j$ for all distinct $i, j \in \mathbf{N}$. Hence there exists $k \in \mathbf{N}$ such that $Rm^k \notin \{M, J\}$. Observe that $M - J - Rm^k - M$

is a cycle of length 3 in $H(R)$.

This completes the proof of Lemma 2.3. □

Corollary 2.4. Let M be a maximal ideal of a ring R such that M is not a B-prime of (0) in R. Then no vertex of $H(R)$ admits a complement in $H(R)$.

Proof. We know from Lemma 2.3 that any edge of $H(R)$ is an edge of a triangle in $H(R)$ and so no vertex of $H(R)$ admits a complement in $H(R)$. □

Proposition 2.5. Let M be a maximal ideal of a ring R such that M is not a B-prime of (0) in R. Then $girth(H(R)) = 3$.

Proof. We first show that $H(R)$ admits at least one edge. Since R is not a field, it follows that $M \neq (0)$. Let $m \in M, m \neq 0$. As M is not a B-prime of (0) in R, we obtain that $Mm \neq (0)$. If $M \neq Rm$, then $M - Rm$ is an edge of $H(R)$. If $M = Rm$, then it is noted in the proof of Case(ii) of Lemma 2.3 that $m \notin Z(R)$. In such a case, $Rm^i \neq Rm^j$ and moreover, $Rm^i - Rm^j$ is an edge of $H(R)$ for all distinct $i, j \in \mathbf{N}$. This shows that $H(R)$ admits at least one edge. Therefore, we obtain from Lemma 2.3 that $girth(H(R)) = 3$. □

3 R is quasilocal

Let R be a quasilocal ring with M as its unique maximal ideal. The aim of this section is to prove some results on $H(R)$ regarding its connectedness, its girth, and determination of its vertices which admit a complement.

Proposition 3.1. Let R be a quasilocal ring with M as its unique maximal ideal such that $H(R)$ has at least two vertices. Then $H(R)$ is connected if and only if M is not a B-prime of (0) in R.

Proof. Assume that $H(R)$ is connected. Suppose that M is a B-prime of (0) in R. Hence there exists $x \in R\backslash\{0\}$ such that $M = ((0) :_R x)$. Since $M \neq (0)$, it follows that $x \in M$. Let A be any proper ideal of R. As $A \subseteq M$, we obtain that $Ax = (0)$. Since we are assuming that $H(R)$ has at least two vertices, there exists a nonzero proper ideal I of R such that $I \neq Rx$. It is clear from the above given arguments that there exists no path in $H(R)$ between I and Rx. This is a contradiction and so M is not a B-prime of (0) in R.

Conversely, assume that M is not a B-prime of (0) in R. Then it follows from Lemma 2.2 that $H(R)$ is connected. □

Let R be a quasilocal ring with M as its unique maximal ideal. In Remark 3.2, we provide information on $diam(H(R))$, in the case when $H(R)$ is connected,

Remark 3.2. If $H(R)$ is connected for a quasilocal ring R, then $diam(H(R)) = 2$ if and only if R is not an integral domain.

Proof. Let T be any integral domain which is not a field. Then T has an infinite number of nonzero proper ideals. Moreover, $H(T) = G(T)$, and for any nonzero ideals I, J of T, $IJ \neq (0)$. Therefore, $diam(H(T)) = 1$.

Let M be the unique maximal ideal of R. Assume that R is not an integral domain and $H(R)$ is connected. Then we know from Proposition 3.1 that M is not a B-prime of (0) in R. Moreover, it follows from Lemma 2.2 that $diam(H(R)) \leq 2$. We next verify that $diam(H(R)) \geq 2$. Since R is not an integral domain, there exist $a, b \in R\backslash\{0\}$ such that $ab = 0$. If $Ra \neq Rb$, then from $ab = 0$, we obtain that Ra and Rb are not adjacent in $H(R)$. Suppose that $Ra = Rb$. Then $a^2 = 0$. As M is not a B-prime of (0) in R, it follows that $Ma \neq (0)$. Let $I = Ma$ and $J = Ra$. From $a^2 = 0$, we get that $IJ = (0)$. We assert that $I \neq J$. For if $I = J$, then $a \in I$ and so $a = ma$ for some $m \in M$. Hence $(1 - m)a = 0$ and as $1 - m$ is a unit in R, we obtain that $a = 0$. This is a contradiction and so $I \neq J$. Note that I and J are not adjacent in $H(R)$. This shows that $diam(H(R)) \geq 2$ and so $diam(H(R)) = 2$. □

Proposition 3.3. Let R be a quasilocal ring such that the unique maximal ideal M of R is not a B-prime of (0) in R. Then the following hold:

(i) No vertex of $H(R)$ admits a complement in $H(R)$.

(ii) $girth(H(R)) = 3$.

Proof. (i) This is an immediate consequence of Corollary 2.4.

(ii) This follows immediately from Proposition 2.5. □

Let T be a ring. Recall from [11] that an ideal I of T is said to be an annihilating ideal if there exists $t \in T\backslash\{0\}$ such that $It = (0)$. As in [11], we denote the set of all annihilating ideals of T by $A(T)$ and we denote the set of all nonzero annihilating ideals of T by $A(T)^*$. Recall from [11] that the annihilating ideal graph of T, denoted by $AG(T)$, is an undirected simple graph whose vertex set is $A(T)^*$ and distinct vertices I, J are joined by an edge in this graph if and only if $IJ = (0)$. The interplay between the ring theoretic properties of T and graph theoretic properties of its annihilating ideal graph $AG(T)$ is well investigated in [11,12]. Let $G = (V, E)$ be an undirected simple graph. Recall from [9, Definition 1.1.13] that the complement of G, denoted by G^c is a graph whose vertex set is V and distinct vertices u, v are joined by an edge in G^c if and only if there is no edge joining u and v in G.

Let R be a quasilocal ring with M as its unique maximal ideal. Suppose that M is a B-prime of (0) in R. That is, there exists $x \in R\backslash\{0\}$ such that $M = ((0) :_R x)$. As each proper ideal of R is contained in M, it follows that the set of all nonzero proper ideals of R equals $A(R)^*$. Moreover, observe that $H(R) = (AG(R))^c$. Hence Proposition 3.4 ifollows immediately from [21,, Lemma 3.3].

Proposition 3.4. Let R be a quasilocal ring with M as its unique maximal ideal. If M is a B-prime of (0) in R, then the following hold:
(i) $H(R)$ is not complemented.
(ii) If a nonzero proper ideal A of R admits a complement in $H(R)$, then so does M.
(iii) If M admits a complement in $H(R)$, then M must be principal. Moreover, M is nilpotent if $\cap_{n=1}^{\infty} M^n = (0)$ and the least positive integer n such that $M^n = (0)$ is at least 4.
(v) If M is not principal, then each edge of $H(R)$ is an edge of a triangle in $H(R)$ and so no vertex of $H(R)$ admits a complement in $H(R)$.
(v) If M is not principal and if $H(R)$ contains at least one edge, then $H(R)$ is a union of triangles and so $girth(H(R)) = 3$.

We conclude this section with some examples to illustrate the results proved in this section.

Example 3.6. The example to be presented here is from [16, Exercises 6 and 7, pp. 62-63]. Let $S = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. Let $M = SX + SY$ and let $T = S_M$. Note that T is a local domain with MT as its unique maximal ideal. Moreover, T is a unique factorization domain admitting an infinite number of nonassociate prime elements. Let $W = \oplus(T/Tp)$ be the direct sum of the T-modules T/Tp, where p varies over all nonassociate prime elements of T. Let $R = T \oplus W$ be the ring obtained on using Nagata's principle of idealization. It is not hard to show that R is quasilocal with $MT \oplus W$ as its unique maximal ideal. It was verified in [22, Example 2.8] that $MT \oplus W$ is not a B-prime of the zero ideal in R. Indeed, it was observed in [2, Example 2.6] that for any nonassociate prime elements p, q of T, there is no nonzero element of R which annihilates both $(p, 0)$ and $(q, 0)$ in R. We obtain from Proposition 3.1 that $H(R)$ is connected and as R is not an integral domain, it follows from Remark 3.2 that $diam(H(R)) = 2$. Moreover, we obtain from Proposition 3.3 that no vertex of $H(R)$ admits a complement in $H(R)$ and $girth(H(R)) = 3$. □

Example 3.7. Let $S = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K.

Let $I = SX^2 + SY^2$. Let $R = S/I$. Let $N = SX + SY$. Note that R is local with $M = N/I$ as its unique maximal ideal. Moreover, $M = ((0+I) :_R XY + I)$ is a B-prime of the zero ideal in R. Observe that $A = R(X+I)$ and $B = R(Y+I)$ are distinct vertices of $H(R)$ and as $XY \notin I$, there is an edge of $H(R)$ joining A and B. From Proposition 3.1, we obtain that $H(R)$ is not connected. Since M is not principal, we obtain from Proposition 3.4(iv) and (v) that no vertex of $H(R)$ admits a complement in $H(R)$ and moreover, $girth(H(R)) = 3$. □

4 R has exactly two maximal ideals

In this section, we consider rings R such that R has exactly two maximal ideals and study some graph theoretic properties of $H(R)$. Let $\{M_1, M_2\}$ denote the set of all maximal ideals of R. In Lemma 4.2, we describe $H(R)$ under the assumption that $M_1 \cap M_2 = (0)$. We classify in Theorem 4.12 rings R such that $H(R)$ is complemented and in Theorem 4.13, we determine necessary and sufficient conditions in order that $H(R)$ contains a cycle. The proof of many results of this section follow closely the proof of results presented in Section 4 of [21].

Lemma 4.1. Let F_1, F_2 be fields and $T = F_1 \times F_2$. Then $H(T)$ is a graph on two vertices and has no edges.

Proof. Note that $\{(0) \times F_2, F_1 \times (0)\}$ is the set of all nonzero proper ideals of T and moreover, $((0) \times F_2)(F_1 \times (0)) = (0) \times (0)$. Hence we obtain the required conclusion. □

Lemma 4.2. Let R be a ring with exactly two maximal ideals and let them be M_1 and M_2. If $M_1 \cap M_2 = (0)$, then $H(R)$ is a graph on two vertices and has no edges.

Proof. Observe that $M_1 + M_2 = R$. As $M_1 \cap M_2 = (0)$, it follows from the Chinese remainder theorem [7, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to R/M_1 \times R/M_2$ defined by $f(r) = (r + M_1, r + M_2)$ is an isomorphism of rings. Let $F_i = R/M_i$ for each $i \in \{1, 2\}$. Note that F_i is a field for each $i \in \{1, 2\}$ and since $R \cong F_1 \times F_2$ as rings, the desired conclusion follows from Lemma 4.1. □

In Propositon 4.4, we determine when $H(R)$ is connected and we make use of Lemma 4.3 in its proof.

Lemma 4.3. Let R be a ring with exactly two maximal ideals and let them be M_1 and M_2. If $M_1 \cap M_2 \neq (0)$, then there exist $a \in M_1 \backslash M_2$ and $b \in M_2 \backslash M_1$ such that $a + b$ is a unit in R and $ab \neq 0$.

Proof. First note that for any $a \in M_1 \backslash M_2$ and $b \in M_2 \backslash M_1$, $a + b \notin M_1 \cup M_2$ = the set of all nonunits in R. Hence $a + b$ is a unit in R. Observe that $M_1 + M_2 = R$. Hence there exist $x \in M_1$ and $y \in M_2$ such that $x + y = 1$. It is clear that $x \notin M_2$ and $y \notin M_1$. If $xy \neq 0$, then the elements $a = x$ and $b = y$ yield the desired conclusion. Suppose that $xy = 0$. Let $z \in M_1 \cap M_2, z \neq 0$. Then from $z = xz + yz$, it follows that either $xz \neq 0$ or $yz \neq 0$. Suppose that $xz \neq 0$. Let $a = x$ and $b = y+z$. It is clear that $a \in M_1 \backslash M_2$, $b \in M_2 \backslash M_1$, and $ab = xz \neq 0$. Moreover, $a + b$ is a unit in R. If $xz = 0$, then $yz \neq 0$. In such a case, the elements $a = x + z$ and $b = y$ satisfy our requirement. □

Proposition 4.4. Let $\{M_1, M_2\}$ denote the set of all maximal ideals of a ring R. Then $H(R)$ is connected if and only if $M_1 \cap M_2 \neq (0)$.

Proof. Assume that $H(R)$ is connected. Then it follows from Lemma 4.2 that $M_1 \cap M_2 \neq (0)$.

Conversely, assume that $M_1 \cap M_2 \neq (0)$. Let I, J be distinct nonzero proper ideals of R. We assert that there exists a path of length at most three in $H(R)$ between I and J. If $IJ \neq (0)$, then $I - J$ is an edge of $H(R)$. Hence we may assume that $IJ = (0)$. We know from Lemma 4.3 that there exist elements $a \in M_1 \backslash M_2$, $b \in M_2 \backslash M_1$ such that $ab \neq 0$. Since $a + b$ is a unit in R, it follows that for any nonzero ideal A of R, either $Aa \neq (0)$ or $Ab \neq (0)$. Suppose that there exists $x \in \{a, b\}$ such that $Ix \neq (0)$ and $Jx \neq (0)$. Then $I - Rx - J$ is a path of length 2 in $H(R)$ between I and J. It may happen that there exists no $x \in \{a, b\}$ such that $Ix \neq (0)$ and $Jx \neq (0)$. We may assume without loss of generality that $Ia \neq (0)$, $Ib = (0)$, $Ja = (0)$, and $Jb \neq (0)$. In such a case, as $ab \neq 0$, it follows that $I - Ra - Rb - J$ is a path of length 3 in $H(R)$ between I and J. This proves that $H(R)$ is connected. □

Let R, M_1, M_2 be as in the statement of Proposition 4.4. With the assumption that $H(R)$ is connected, in Proposition 4.5, we determine $diam(H(R))$.

Proposition 4.5. Let $\{M_1, M_2\}$ denote the set of all maximal ideals of a ring R. Suppose that R is not an integral domain. If $H(R)$ is connected, then $2 \leq diam(H(R)) \leq 3$. Moreover, $diam(H(R)) = 3$ if and only if both M_1 and M_2 are B-primes of (0) in R.

Proof. Assume that $H(R)$ is connected. Then $M_1 \cap M_2 \neq (0)$ and it is shown in the proof of Proposition 4.4, that for any distinct vertices I, J of $H(R)$, there exists a path of length at most 3 in $H(R)$ between I and J. This shows that $diam(H(R)) \leq 3$. We next show that there exist nonzero proper ideals A, B of R such that A and B are not adjacent in $H(R)$, that is $AB = (0)$. As we are assuming that R is not an integral domain, there exist $x, y \in R \backslash \{0\}$ such that $xy = 0$. If $Rx \neq Ry$, then the ideals $A = Rx$ and $B = Ry$ are distinct vertices of $H(R)$ and $AB = (0)$. Suppose that $Rx = Ry$. Then $x^2 = 0$. We know from Lemma 4.3 that there exist $a \in M_1 \backslash M_2$ and $b \in M_2 \backslash M_1$ such that $ab \neq 0$. As $a + b$ is a unit in R, either $ax \neq 0$ or $bx \neq 0$. Without loss of generality, we may assume that $ax \neq 0$. If $abx = 0$, then $A = Rax$, $B = Rb$ are distinct vertices of $H(R)$ and $AB = (0)$. Suppose that $abx \neq 0$. Let $A = Rabx$ and $B = Rx$. It follows from $x^2 = 0$ that $AB = (0)$. We claim that $A \neq B$. For if $A = B$, then $x = rabx$ for some $r \in R$. This implies that $(1 - rab)x = 0$. Since $ab \in M_1 \cap M_2$, $1 - rab \notin M_1 \cup M_2$. Hence $1 - rab$ is a unit in R and so it follows from $(1 - rab)x = 0$ that $x = 0$. This is a contradiction and therefore, $Rabx \neq Rx$. This shows that $diam(H(R)) \geq 2$. Thus $2 \leq diam(H(R)) \leq 3$.

Suppose that either M_1 or M_2 is not a B-prime of (0) in R. Then it follows from Lemma 2.2 that $diam(H(R)) \leq 2$ and so $diam(H(R)) = 2$. Suppose that both M_1 and M_2 are B-primes of (0) in R. Hence there exist $a_1, a_2 \in R \backslash \{0\}$ such that $M_i = ((0) :_R a_i)$ for each $i \in \{1, 2\}$. It is clear that $Ra_1 \neq Ra_2$ and we know from [10, Lemma 3.6] that $a_1 a_2 = 0$. Thus Ra_1 and Ra_2 are not adjacent in $H(R)$. Let A be any proper nonzero ideal of R. Then either $Aa_1 = (0)$ or $Aa_2 = (0)$. Hence there exists no vertex A of $H(R)$ which is adjacent to both Ra_1 and Ra_2 in $H(R)$. This proves that $d(Ra_1, Ra_2) \geq 3$ in $H(R)$ and therefore, $diam(H(R)) = 3$. □

Remark 4.6. Let R, M_1, M_2 be as in the statement of Proposition 4.4. We proceed to determine necessary and sufficient conditions in order that $H(R)$ contains a cycle and moreover, our goal is to classify rings R such that $H(R)$ is complemented. We need several lemmas to arrive at the desired results. Let $I(R)$ denote the set of all proper ideals of R and let $I(R)^* = I(R) \backslash \{(0)\}$. Let $\mathcal{A} = \{I \in I(R)^* | I \subseteq M_1, I \nsubseteq M_2\}$ and let $\mathcal{B} = \{J \in I(R)^* | J \subseteq M_2, J \nsubseteq M_1\}$. Observe that $M_1 \in \mathcal{A}$ and $M_2 \in \mathcal{B}$. □

Lemma 4.7. Let $\{M_1, M_2\}$ denote the set of all maximal ideals of a ring R. Suppose that for each $i \in \{1, 2\}$, $M_i = ((0) :_R a_i)$ for some $a_i \in R \backslash \{0\}$. If $M_1 \cap M_2 \neq (0)$, then either $a_1 \in M_1 \cap M_2$ or $a_2 \in M_1 \cap M_2$.

Proof. We know from [10, Lemma 3.6] that $a_1 a_2 = 0$. Thus $a_1 \in M_2$ and $a_2 \in M_1$. Suppoe that $a_i \notin M_1 \cap M_2$ for each $i \in \{1, 2\}$. Then $a_1 \in M_2 \backslash M_1$ and $a_2 \in M_1 \backslash M_2$. Therefore, $a_1 + a_2$ is a unit in R. Let $x \in M_1 \cap M_2$, $x \neq 0$. Note that $(a_1 + a_2)x = 0$ and this implies that $x = 0$. This is a contradiction and hence we obtain the desired conclusion. □

Lemma 4.8. Let R, M_1, M_2 be as in the statement of Proposition 4.4. Suppose that $M_1 \cap M_2 \neq (0)$. Let $\mathcal{A}$, $\mathcal{B}$ be as in Remark 4.6. If $|\mathcal{A}| \geq 2$ and $|\mathcal{B}| \geq 2$, then the following hold:

(i) $H(R)$ is not complemented.

(ii) $girth(H(R)) = 3$.

Proof. If either M_1 or M_2 is not a B-prime of (0) in R, then we know from Corollary 2.4 that no vertex of $H(R)$ admits a complement in $H(R)$ and moreover, we know from Proposition 2.5 that $girth(H(R)) = 3$. Hence in proving (i) and (ii) of this lemma, we may assume that both M_1 and M_2 are B-primes of (0) in R. Let $a_i \in R\backslash\{0\}$ be such that $M_i = ((0) :_R a_i)$ for each $i \in \{1, 2\}$.

(i) Since we are assuming that $M_1 \cap M_2 \neq (0)$, we know from Lemma 4.7 that either $a_1 \in M_1 \cap M_2$ or $a_2 \in M_1 \cap M_2$. Without loss of generality, we may assume that $a_1 \in M_1 \cap M_2$. We claim that Ra_1 does not admit a complement in $H(R)$. Suppose that there exists a nonzero proper ideal I of R such that $Ra_1 \perp I$ in $H(R)$. From $Ia_1 \neq (0)$, it follows that $I \not\subseteq M_1$. As $I \neq R$, $I \subseteq M_2$. Therefore, $I \in \mathcal{B}$. By hypothesis, $|\mathcal{B}| \geq 2$. Let $J \in \mathcal{B}$ be such that $J \neq I$. Note that $J \notin \{Ra_1, I\}$. As $I \not\subseteq M_1$ and $J \not\subseteq M_1$, it follows that $IJ \not\subseteq M_1$ and so $IJ \neq (0)$. It is clear that $Ja_1 \neq (0)$. Hence J is adjacent to both Ra_1 and I in $H(R)$. This is in contradiction to the assumption that $Ra_1 \perp I$ in $H(R)$. Hence Ra_1 does not admit a complement in $H(R)$ and so $H(R)$ is not complemented.

(ii) As in (i), we may assume without loss of generality that $a_1 \in M_1 \cap M_2$. Let J_1, J_2 be any distinct members of $\mathcal{B}$. It is clear that $Ra_1 \notin \{J_1, J_2\}$, $J_k a_1 \neq (0)$ for each $k \in \{1, 2\}$, and $J_1 J_2 \neq (0)$. Therefore, $J_1 - Ra_1 - J_2 - J_1$ is a cycle of length 3 in $H(R)$ and so $girth(H(R)) = 3$. $\square$

Lemma 4.9. Let $\{M_1, M_2\}$ denote the set of all maximal ideals of a ring R. Suppose that $M_1 \cap M_2 \neq (0)$. Let $\mathcal{A}$, $\mathcal{B}$ be as in Remark 4.6. If $|\mathcal{A} \cup \mathcal{B}| \geq 4$, then the following hold:

(i) $H(R)$ is not complemented.

(ii) $girth(H(R)) = 3$.

Proof. In view of Lemma 4.8, we may assume that either $|\mathcal{A}| = 1$ or $|\mathcal{B}| = 1$. Without loss of generality, we may assume that $|\mathcal{A}| = 1$. As we are assuming that $\mathcal{A} \cup \mathcal{B}$ contains at least four elements, it follows that $|\mathcal{B}| \geq 3$.

(i) Observe that $M_2 \in \mathcal{B}$. Let $J \in \mathcal{B}$ be such that $J \neq M_2$. We claim that J does not admit a complement in $H(R)$. Suppose that J admits a complement in $H(R)$. Let I be a nonzero proper ideal of R such that $J \perp I$ in $H(R)$. From $JI \neq (0)$, it follows that $M_2 I \neq (0)$. Note that for any $J_1, J_2 \in \mathcal{B}$, $J_1 J_2 \not\subseteq M_1$ and so $J_1 J_2 \neq (0)$. Hence $M_2 J \neq (0)$. It follows from the assumption $J \perp I$ that $I = M_2$. Since $|\mathcal{B}| \geq 3$, there exists $K \in \mathcal{B}$ such that $K \notin \{J, M_2\}$. The fact that K is adjacent to both J and $I = M_2$ in $H(R)$ contradicts the assumption that $J \perp I$ in $H(R)$. This proves the claim and so $H(R)$ is not complemented.

(ii) Let J_1, J_2, J_3 be any three distinct members of $\mathcal{B}$. Note that $J_1 - J_2 - J_3 - J_1$ is a cycle of length 3 in $H(R)$ and therefore, $girth(H(R)) = 3$. □

Lemma 4.10. Let $\{M_1, M_2\}$ denote the set of all maximal ideals of a ring R. If $(M_1 \cap M_2)^2 \neq (0)$, then the following hold:

(i) $H(R)$ is not complemented.

(ii) $girth(H(R)) = 3$.

Proof. We claim that $M_1 \cap M_2$ does not admit a complement in $H(R)$. Suppose that there exists a nonzero proper ideal I of R such that $M_1 \cap M_2 \perp I$ in $H(R)$. Observe that either $I \neq M_1$ or $I \neq M_2$. Without loss of generality, we may assume that $I \neq M_1$. From $(M_1 \cap M_2)I \neq (0)$ and $(M_1 \cap M_2)^2 \neq (0)$, it follows that $M_1 I \neq (0)$ and $M_1(M_1 \cap M_2) \neq (0)$. Hence M_1 is adjacent to both $M_1 \cap M_2$ and I in $H(R)$. This contradicts the assumption that $M_1 \cap M_2 \perp I$ in $H(R)$. This proves the claim and so $H(R)$ is not complemented.

(ii) It follows from $(M_1 \cap M_2)^2 \neq (0)$ that $M_1 - M_1 \cap M_2 - M_2 - M_1$ is a cycle of length 3 in $H(R)$. Hence $girth(H(R)) = 3$. □

Recall that a principal ideal ring T is called a special principal ideal ring (SPIR) if it has a unique prime ideal. If M denotes the unique prime ideal of a SPIR T, then M is necessarily nilpotent. If $n \geq 2$ is least with the property that $M^n = (0)$, then it follows from $(iii) \Rightarrow (i)$ of [7, Proposition 8.8] that $\{M, \ldots, M^{n-1}\}$ is the set of all nonzero proper ideals of T. If T is a SPIR with M as its unique prime ideal, then we denote it using the notation (T, M) is a SPIR.

Lemma 4.11. Let T_1 be a nonzero ring and let T_2 be a ring with a prime ideal P such that $P \neq (0)$. Let $T = T_1 \times T_2$. Then $|I(T)^*| = 4$ if and only if T_1 is a field and (T_2, P) is a SPIR with $P^2 = (0)$.

Proof. Assume that T has exactly 4 nonzero proper ideals, that is $|I(T)^*| = 4$. In such a case, it is clear that $I(T)^* = \{(0) \times P, (0) \times T_2, T_1 \times (0), T_1 \times P\}$. Let I be any nonzero ideal of T_1. Then $I \times (0) \in I(T)^*$ and so $I \times (0) = T_1 \times (0)$. This implies that $I = T_1$. This proves that T_1 is a field. Let J be any nonzero proper ideal of T_2. Then $(0) \times J \in I(T)^*$ and so $(0) \times J = (0) \times P$. Hence $J = P$. This shows that P is the only nonzero proper ideal of T_2. Let $p \in P, p \neq 0$. Then $P = T_2 p$. It is clear that $P^2 \subset P$ and so $P^2 = (0)$. Hence (T_2, P) is a SPIR with $P^2 = (0)$.

Conversely, assume that T_1 is a field and (T_2, P) is a SPIR with $P^2 = (0)$. Then (0)

is the only proper ideal of T_1 and $\{(0), P\}$ is the set of all proper ideals of T_2. Therefore, $\{(0)\times P, (0)\times T_2, T_1\times(0), T_1\times P\}$ is the set of all nonzero proper ideals of T. Hence $|I(T)^*| = 4$. $\square$

Let $G = (V, E)$ be a connected graph with at least two vertices. If G does not contain any triangle, then it is clear that G is complemented. For a ring R with exactly two maximal ideals, if $H(R)$ is complemented, then we verify in Theorem 4.12 that $H(R)$ does not contain any triangle and moreover, in Theorem 4.12, we classify rings R such that $H(R)$ is complemented.

Theorem 4.12. Let $\{M_1, M_2\}$ denote the set of all maximal ideals of a ring R. Then $H(R)$ is complemented if and only if $R \cong F \times S$ as rings, where F is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$.

Proof. Assume that $H(R)$ is complemented. Then it follows from Lemma 4.2 that $M_1 \cap M_2 \neq (0)$. Let $\mathcal{A}$ and $\mathcal{B}$ be as in Remark 4.6. Note that $M_1 \in \mathcal{A}$ and $M_2 \in \mathcal{B}$. It follows from Lemma 4.9(i) that $|\mathcal{A} \cup \mathcal{B}| \leq 3$. Without loss of generality, we may assume that $|\mathcal{A}| = 1$ and $|\mathcal{B}| \leq 2$. Observe that $M_1, M_1^2 \in \mathcal{A}$ and as $|\mathcal{A}| = 1$, it follows that $M_1 = M_1^2$. Moreover, we obtain from Lemma 4.10(i) that $(M_1 \cap M_2)^2 = (0)$. Hence $M_1M_2^2 = M_1^2M_2^2 = (0)$. As $M_1 + M_2 = R$, it follows that $M_1M_2 = M_1 \cap M_2$ and so $M_1M_2 \neq (0)$. Since $M_1M_2^2 = (0)$, we obtain that $M_2 \neq M_2^2$. It follows from $M_2, M_2^2 \in \mathcal{B}$ and $|\mathcal{B}| \leq 2$ that $\mathcal{B} = \{M_2, M_2^2\}$. It can be shown as in the proof of [21, Theorem 4.10] that if I is any nonzero ideal of R such that $I \subseteq M_1 \cap M_2$, then $I = M_1 \cap M_2$. As $I(R)^* = \mathcal{A} \cup \mathcal{B} \cup \{M_1 \cap M_2\}$, we obtain that $I(R)^* = \{M_1, M_2, M_2^2, M_1 \cap M_2\}$.

Since $M_1 + M_2^2 = R$ and $M_1M_2^2 = (0)$, we obtain from the Chinese remainder theorem that the mapping $f : R \to R/M_1 \times R/M_2^2$ defined by $f(r) = (r + M_1, r + M_2^2)$ is an isomorphism of rings. Let $T = R/M_1 \times R/M_2^2$. As $R \cong T$ as rings, it follows that $|I(T)^*| = 4$. Hence we obtain from Lemma 4.11 that $(R/M_2^2, M_2/M_2^2)$ is a SPIR. Let $F = R/M_1$, $S = R/M_2^2$, and $M = M_2/M_2^2$. Note that F is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$ and $R \cong F \times S$ as rings.

Conversely, assume that $R \cong F \times S$ as rings, where F is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$. Let $T = F \times S$. Note that $H(T)$ is a graph on the vertex set $\{(0)\times M, (0)\times S, F\times(0), F\times M\}$. Note that $(0)\times M \perp (0)\times S$ and $F\times(0) \perp F\times M$ in $H(T)$. This proves that $H(T)$ is complemented. Indeed, $H(T)$ is the path $(0)\times M - (0)\times S - F\times M - F\times(0)$. As $R \cong T$ as rings, we obtain that $H(R)$ is complemented. Moreover, observe that each vertex of $H(R)$ admits a unique complement in $H(R)$. $\square$

Let R, M_1, M_2 be as in the statement of Theorem 4.12. In Theorem 4.13, we determine necessary and sufficient conditions in order that $H(R)$ contains a cycle.

Theorem 4.13. Let $\{M_1, M_2\}$ denote the set of all maximal ideals of a ring R. Then the following statements are equivalent:

(i) $H(R)$ contains a cycle.

(ii) $M_1 \cap M_2 \neq (0)$ and moreover, R is not isomorphic to $F \times S$ as rings, where F is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$.

(iii) $girth(H(R)) = 3$.

Proof. $(i) \Rightarrow (ii)$ Assume that $H(R)$ contains a cycle. Then it follows from Lemma 4.2 that $M_1 \cap M_2 \neq (0)$. Suppose that $R \cong F \times S$ as rings, where F is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$. Let $T = F \times S$. It is already noted in the proof of Theorem 4.12 that $H(T)$ is the path $(0) \times M - (0) \times S - F \times M - F \times (0)$. Thus $H(T)$ does not contain any cycle and so $H(R)$ does not contain any cycle. This is a contradiction. This proves $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (iii)$ Let $\mathcal{A}$ and $\mathcal{B}$ be as in Remark 4.6. Observe that $M_1 \in \mathcal{A}$ and $M_2 \in \mathcal{B}$. Suppose that $girth(H(R)) \neq 3$. It follows from Lemma 4.9(ii) that $|\mathcal{A} \cup \mathcal{B}| \leq 3$. Without loss of generality, we may assume that $|\mathcal{A}| = 1$ and $|\mathcal{B}| \leq 2$. Observe that as is remarked in the proof of Theorem 4.12, $M_1 = M_1^2$. Moreover, we obtain from Lemma 4.10(ii) that $(M_1 \cap M_2)^2 = (0)$. Now proceeding as in the proof of Theorem 4.12, we obtain that $M_2 \neq M_2^2$ and $R \cong R/M_1 \times R/M_2^2$ as rings with $(R/M_2^2, M_2/M_2^2)$ is a SPIR. This contradicts (ii). This proves $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (i)$ This is clear. □

Remark 4.14. Let $\{M_1, M_2\}$ denote the set of all maximal ideals of a ring R. It is proved in Theorem 4.12 that $H(R)$ is complemented if and only if $R \cong F \times S$ as rings, where F is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$. As each proper ideal of $F \times S$ is an annihilating ideal, it follows that $I(R)^* = A(R)^*$. Hence $H(R) = (AG(R))^c$.

If $M_1 \cap M_2 \neq (0)$, then it follows from Theorem 4.13 that $girth(H(R)) \neq 3$ if and only if $R \cong F \times S$ as rings, where F is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$. As is remarked in the previous paragraph, we obtain that $H(R) = (AG(R))^c$. □

We provide some examples to illustrate the results proved in this section.

Example 4.15.(i) Let R_1 be a quasilocal ring with M_1 as its unique maximal ideal such that M_1 is not a B-prime of (0) in R_1 (one can consider R_1 equals the quasilocal ring R as in Example 3.6). Let R_2 be a quasilocal ring with M_2 as its unique maximal ideal such that M_2 is a B-prime of (0) in R_2 (one can consider R_2 equals the local ring R as in Example 3.7). Let

$R = R_1 \times R_2$. Note that $\{N_1 = M_1 \times R_2, N_2 = R_1 \times M_2\}$ is the set of all maximal ideals of R and $N_1 \cap N_2 = M_1 \times M_2 \neq (0) \times (0)$. Since M_1 is not a B-prime of (0) in R_1, it follows that N_1 is not a B-prime of the zero ideal in R. It follows from Propositions 4.4 and 4.5 that $H(R)$ is connected and moreover, $diam(H(R)) = 2$.

(ii) Let $T = R \times R$, where R is the local ring considered in Example 3.7. It was noted in Example 3.7 that the unique maximal M of R is a B-prime of the zero ideal in R. Observe that $\{M_1 = M \times R, M_2 = R \times M\}$ is the set of all maximal ideals of T, $M_1 \cap M_2 = M \times M \neq (0) \times (0)$, and moreover, as M is a B-prime of the zero ideal in R, it follows that both M_1 and M_2 are B-primes of the zero ideal in T. Hence we obtain from Propositions 4.4 and 4.5 that $H(T)$ is connected and moreover, $diam(H(T)) = 3$.

5 R has more than two maximal ideals

In this section, we consider rings R such that R has at least three maximal ideals and investigate some properties of $H(R)$.

Proposition 5.1. Let R be a ring which admits at least three maximal ideals. Then $H(R)$ is commected and $diam(H(R)) \leq 2$. Moreover, if R is not an integral domain, then $diam(H(R)) = 2$.

Proof. Let I, J be distinct nonzero proper ideals of R. If $IJ \neq (0)$, then $I - J$ is a path of length 1 in $H(R)$ between I and J. Suppose that $IJ = (0)$. Since R has at least three maximal ideals, there exists a maximal ideal M of R such that $M \neq ((0) :_R I)$ and $M \neq ((0) :_R J)$. Hence $M \not\subseteq ((0) :_R I) \cup ((0) :_R J)$. Therefore, there exists $m \in M$ such that $Im \neq (0)$ and $Jm \neq (0)$. Note that $I - Rm - J$ is a path of length 2 in $H(R)$ between I and J. This proves that $H(R)$ is connected and $diam(H(R)) \leq 2$.

Assume that R is not an integral domain. Hence there exists $a, b \in R\backslash\{0\}$ such that $ab = 0$. Let $I = Ra$ and $J = Rb$. Observe that $IJ = (0)$. If $I \neq J$, then I and J are not adjacent in $H(R)$. Suppose that $I = J$. Then $a^2 = 0$. Let M_1, M_2, M_3 be any three distinct maximal ideals of R. As $a \in M_3$ but $M_1 \cap M_2 \not\subseteq M_3$, it follows that $M_1 \cap M_2 \neq Ra$. If $(M_1 \cap M_2)a = (0)$, then $M_1 \cap M_2$ and Ra are not adjacent in $H(R)$. Suppose that $(M_1 \cap M_2)a \neq (0)$. If $(M_1 \cap M_2)a \neq Ra$, it follows from $a^2 = 0$ that $(M_1 \cap M_2)a$ and Ra are not adjacent in $H(R)$. Suppose that $(M_1 \cap M_2)a = Ra$. Then $a = xa$ for some $x \in M_1 \cap M_2$. This implies that $(1 - x)a = 0$. Note that $1 - x$ is a nonzero nonunit of R. Since $1 - x \notin M_1 \cap M_2$ and $a \in M_1 \cap M_2$, it follows that $R(1 - x) \neq Ra$. From $(1 - x)a = 0$, we obtain that $R(1 - x)$ and Ra are not adjacent in $H(R)$.

Thus there exist nonzero proper ideals A, B of R such that A and B are not adjacent in $H(R)$. Therefore, $diam(H(R)) \geq 2$ and so $diam(H(R)) = 2$. □

Proposition 5.2. Let R be a ring which admits at least three maximal ideals. Then any edge of $H(R)$ is an edge of a triangle in $H(R)$. Moreover, $girth(H(R)) = 3$.

Proof. Let $I - J$ be an edge of $H(R)$. We consider the following cases:

Case(i). $I + J \subseteq M$ for some maximal ideal M of R

Since we are assuming that R has at least three maximal ideals, there exists a maximal ideal N of R such that $N \neq M$ and $N \neq ((0) :_R IJ)$. Hence $N \not\subseteq M \cup ((0) :_R IJ)$. Therefore, there exists $x \in N$ such that $x \notin M$ and $IJx \neq (0)$. As $x \notin M$, it is clear that $Rx \notin \{I, J\}$ and moreover, note that $I - J - Rx - I$ is a cycle of length 3 in $H(R)$.

Case(ii) $I + J = R$

Let M_1 be a maximal ideal of R such that $I \subseteq M_1$. Let M_2 be a maximal ideal of R such that $J \subseteq M_2$. Since $I + J = R$, we obtain that $M_1 \neq M_2$. Observe that $I((0) :_R I) = (0)$ and as $I \not\subseteq M_2$, it follows that $((0) :_R I) \subseteq M_2$. Similarly, as $J \not\subseteq M_1$, we obtain from $J((0) :_R J) = (0)$ that $((0) :_R J) \subseteq M_1$. Let M_3 be a maximal ideal of R such that $M_3 \notin \{M_1, M_2\}$. Since $M_3 \not\subseteq M_1 \cup M_2$, it is clear that $M_3 \notin \{I, J\}$ and $M_3 I \neq (0)$ and $M_3 J \neq (0)$. Therefore, $I - J - M_3 - I$ is a cycle of length 3 in $H(R)$.

Note that if M_1, M_2, M_3 are three distinct maximal ideals of R, then $M_1 - M_2 - M_3 - M_1$ is a cycle of length 3 in $H(R)$. Hence $girth(H(R)) = 3$. □

Corollary 5.3. Let R be a ring which admits at least three maximal ideals. Then no vertex of $H(R)$ admits a complement in $H(R)$.

Proof. We know from Proposition 5.2 that each edge of $H(R)$ is an edge of a triangle in $H(R)$. Therefore, no vertex of $H(R)$ admits a complement in $H(R)$. □

We provide some examples to illustrate the results proved in this section.

Example 5.4. (i) Let $n \in \mathbf{N}$ be such that n has at least three distinct prime divisors. Let $R = \mathbf{Z}/n\mathbf{Z}$. As n has at least three distinct prime divisors, it follows that R has at least three maximal ideals. It follows from Proposition 5.1 that $H(R)$ is connected and moreover, $diam(H(R)) = 2$. Furthermore, we obtain from Proposition 5.2 that any edge of $H(R)$ is an edge of a triangle in $H(R)$ and hence we get that no vertex of $H(R)$ admits a complement in $H(R)$.

(*ii*) Let $m \geq 3$ and let $T = R^m$ be the direct product of m copies of R, where R is the quasilocal ring considered in Example 3.6. Observe that T has exactly $m \geq 3$ maximal ideals. Therefore, we obtain from Propositions 5.1 and 5.2 that $H(T)$ is connected, $diam(H(T)) = 2$, and any edge of $H(T)$ is an edge of a triangle in $H(T)$. Hence it follows that no vertex of $H(T)$ admits a complement in $H(T)$. Let $T_1 = R[X]$ be the polynomial ring in one variable X over R. Then as T_1 has infinitely many maximal ideals, we obtain that $H(T_1)$ also has the same properties as that of $H(T)$.

6 Concluding Remarks

Let R be a ring which is not a field. For the sake of easy reference and convenience, in this Section, we mention the properties of $H(R)$ that are proved in Sections 3,4 and 5 of this article.

Proposition 6.1. Let R be a ring such that R has at least two nonzero proper ideals. Then $H(R)$ is connected if and only if either $(a), (b)$ or (c) holds:
(a) R is quasilocal with M as its unique maximal such that M is not a B-prime of (0) in R.
(b) R has exactly two maximal ideals M_1, M_2 with $M_1 \cap M_2 \neq (0)$.
(c) R has at least three maximal ideals.

Moreover, if (a) or (c) holds, then $diam(H(R)) \leq 2$ and $diam(H(R)) = 2$ if and only if R is not an integral domain. If (b) holds and if R is not an integral domain, then $2 \leq diam(H(R)) \leq 3$ and $diam(H(R)) = 3$ if and only if both M_1 and M_2 are B-primes of (0) in R.

Proof. The proof of this proposition follows immediately from Propositon 3.1, Remark 3.2, Propositions 4.4, 4.5, and 5.1. □

Proposition 6.2. Let R be a ring which is not a field. Then $girth(H(R)) = 3$ or ∞. Indeed, the following hold:
(i) Suppose that R is quasilocal with M as its unique maximal ideal.
(a) If M is not a B-prime of (0) in R, then $H(R)$ is a union of triangles and hence $girth(H(R)) = 3$.
(b) If M is a B-prime of (0) in R and if M is not principal, then $H(R)$ is a union of triangles if and only if $M^2 \neq (0)$. Thus $girth(H(R)) = 3$ if $M^2 \neq (0)$ and ∞ otherwise.
(c) If M is principal and is not nilpotent, then $girth(H(R)) = 3$.
(d) If M is principal and is nilpotent (that is, equivalently, (R, M) is a SPIR), then $girth(H(R)) = 3$ if $M^5 \neq (0)$ and ∞ otherwise.
(ii) Suppose that R has exactly two maximal ideals M_1, M_2.

(a) If $M_1 \cap M_2 = (0)$, then $girth(H(R)) = \infty$.
(b) If $M_1 \cap M_2 \neq (0)$, then $girth(H(R)) = 3$ if and only if $R \ncong F \times S$ as rings, where F is a field, and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$. Othewise, $girth(H(R)) = \infty$.
(iii) If R has at least three maximal ideals, then $H(R)$ is a union of triangles and hence $girth(H(R)) = 3$.

Proof. (i) (a) This follows immediately from Lemma 2.3 and Proposition 2.5.
(b) We first verify that $H(R)$ has at least one edge if and only if $M^2 \neq (0)$. If $M^2 = (0)$, then for any nonzero proper ideals I, J of R, $IJ = (0)$. Hence $H(R)$ has no edges. If $M^2 \neq (0)$, then $Mm \neq (0)$ for some $m \in M$. By assumption M is not principal and so $M \neq Rm$. From $Mm \neq (0)$, it follows that there is an edge of $H(R)$ joining M and Rm. The proof of (b) follows immediately from Proposition 3.4 (v).
(c) Let $M = Rm$ for some $m \in M, m \neq (0)$. From $m^k \neq 0$ for all $k \in \mathbf{N}$, it follws that $Rm^i \neq Rm^j$ for all distinct $i, j \in \mathbf{N}$. Note that $Rm - Rm^2 - Rm^3 - Rm$ is a cycle of length 3 in $H(R)$ and so $girth(H(R)) = 3$.
(d) Observe that if M is niplotent, then each nonzero proper ideal of R is an annihilating ideal and hence $H(R) = (AG(R))^c$. Therefore, (d) follows from [21, Lemma 6.3].
(ii) (a) It follows from Lemma 4.2 that $girth(H(R)) = \infty$.
(b) This follows from Propsition 4.13.
(iii) This follows from Propsition 5.2. □

Proposition 6.3. Let R be a ring which is not a field. Then $H(R)$ is complemented if and only if $R \cong F \times S$ as rings, where F is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$.

Proof. This follows from Propositions 3.3(i), 3.4(i), Theorem 4.12, and Corollary 5.3. □

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