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Geodesics in Interior Black Hole Region

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ABSTRACT

The metric for the interior black hole region was derived earlier in the frame work of general relativity [4, 5]. A geometric study of the spacetime described by this metric reveals many curvature properties [6]. In this paper geodesic equation has been derived. It is found that the geodesics are similar to the geodesics obtained from Schwarzschild exterior solution with the interchanged role of spatial and temporal coordinates.

Key words: General Relativity, Black Hole, Geodesic

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1. Introduction

The Schwarzschild solution has played a fundamental conceptual role in general relativity and beyond, for instance, regarding event horizons, spacetime singularities and aspects of quantum field theory in curved spacetimes. Before the mid-1960s, the object now known as a black hole, was referred as a collapsed star Thorne et al. [1], as a frozen star Zel'dovich and Novikov [2], and it was only in 1965 that marked an era of intensive research into black hole physics. A black hole is such an object where the spacetime is so warped that a quantum description of gravity becomes essential for underlying spacetime. Black hole physics has been one of the leading areas in theoretical physics since the derivation of the first non-trivial solution of Einstein's equation for a homogeneous and isotropic spacetime, by Karl Schwarzschild in the famous paper [3].

In this work, we derive the geodesic equation and shows that in the interior region of a black hole it is similar to the equation of geodesic obtained in Schwarzschild spacetime but with the interchanged role of spatial and temporal coordinates.

Suppose the physical radius of a star is a. For the case $z < a$, spacetime is no more empty and if $a < 2m$ then the spacetime is divided into three regions:

(1) $z < a$: physical content of the star

(2) $a < z < 2m$: empty region of spacetime

(3) $z > 2m$: empty region of spacetime represented by Schwarzschild exterior metric.

The region $z > 2m$ is represented by Schwarzschild exterior metric

$$ds^2 = \left(1 - \frac{2m}{z}\right)dt^2 - \left(1 - \frac{2m}{z}\right)^{-1} dz^2 - z^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \ldots (1)$$

This solution is the first non-trivial solution of Einstein's field equations for empty space for the simplest case of spherical symmetry. The fact that Schwarzschild solution is not just a good solution, but is the unique spherical symmetric solution, as proved in Birkhoff's theorem. The metric describing spacetime for the region $a < z < 2m$ of a black hole is called the interior black hole solution.

For $z < 2m$, Schwarzschild exterior metric can be written,

$$ds^2 = -\left(\frac{2m}{z} - 1\right)dt^2 + \left(\frac{2m}{z} - 1\right)^{-1} dz^2 - z^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

That is $$ds^2 = \left(\frac{2m}{z} - 1\right)^{-1} dz^2 - \left(\frac{2m}{z} - 1\right)dt^2 - z^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \ldots (2)$$

which is valid for $z < 2m$.

Equation (2) represents the annular region of spacetime with $a < z < 2m$, that is for a black hole. In this region the roles of co-ordinates z and t are interchanged which can be seen as a signature change in the metric given in (2). By assuming a non-static line element as $ds^2 = B(z,t)c^2dt^2 - A(z,t)dz^2 - c^2t^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ the interior black hole metric in the following form is obtained [1]:

$$ds^2 = \left(\frac{2\xi}{t} - 1\right)^{-1} dt^2 - \left(\frac{2\xi}{t} - 1\right)dz^2 - t^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \ldots (3)$$

2. Geodesics in Interior Black Hole Region

For metric (3), the Lagrangian is written as

$$L = 1 = \left(\frac{2\xi}{t} - 1\right)^{-1}\left(\frac{dt}{ds}\right)^2 - \left(\frac{2\xi}{t} - 1\right)\left(\frac{dz}{ds}\right)^2 - t^2\left(\frac{d\theta}{ds}\right)^2 - t^2\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 \;...(4)$$

To find geodesics sometimes applying the Euler-Lagrange equations is easier than the calculations of the coefficients of the Levi-Civita connection.

Consider, Lagrange's equation of motion

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

So, $\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$ gives $\frac{d}{ds}\left[-2A\left(\frac{dz}{ds}\right)\right] = 0$

i.e. $\frac{dz}{ds} = \frac{C_0}{\left(\frac{2\xi}{t} - 1\right)}$ ... (5)

Similarly, $\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$ gives $\frac{d}{ds}\left[-2t^2 \cdot \frac{d\theta}{ds}\right] - 2t^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0$

That has a particular solution $\frac{d\theta}{ds} = 0$.

i.e. θ is constant... (6)

Also $\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$ gives $\frac{d}{ds}\left[-2t^2\sin^2\theta \cdot \frac{d\phi}{ds}\right] = 0$

On integration, we have $\frac{d\phi}{ds} = \frac{h}{t^2\sin^2\theta}$ ... (7)

Alike the case for the geodesics in the Schwarzschild exterior region, consider the motion in the equatorial plane $\theta = \frac{\pi}{2}$ with $\frac{dz}{ds} = \frac{C_0}{\left(\frac{2\xi}{t} - 1\right)}$ and $\frac{d\phi}{ds} = \frac{h}{t^2}$.

If $t = t(\phi)$ with $\theta = \frac{\pi}{2}$ then $\frac{dt}{ds} = \frac{dt}{d\phi} \cdot \frac{d\phi}{ds} = \frac{h}{t^2} \cdot \frac{dt}{d\phi}$

Substituting in (4),

$$1 = -\left(\frac{2\xi}{t} - 1\right)\frac{C_0^2}{\left(\frac{2\xi}{t} - 1\right)^2} - t^2\left(\frac{h^2}{t^4}\right) + \left(\frac{2\xi}{t} - 1\right)^{-1}\left(\frac{h}{t^2}\right)^2\left(\frac{dt}{d\phi}\right)^2$$

$$\text{i.e. } \left(\frac{2\xi}{t} - 1\right) = -C_0^2 - \left(\frac{2\xi}{t} - 1\right)\frac{h^2}{t^2} + \frac{h^2}{t^4}\left(\frac{dt}{d\phi}\right) \dots (8)$$

Choosing new dependent variable $u = \frac{1}{t}$ so that $\frac{du}{d\phi} = -\frac{1}{t^2} \cdot \frac{dt}{d\phi}$

Hence, the equation (8) becomes $(2u\xi - 1) = -C_0^2 - (2u\xi - 1)h^2u^2 + h^2\left(\frac{du}{d\phi}\right)^2$

i.e. $(2u\xi - 1) = h^2\left(\frac{du}{d\phi}\right)^2 - 2u^3\xi h^2 + h^2u^2 - C_0^2$

Differentiating with respect to ϕ, we get

$$2\xi \cdot \frac{du}{d\phi} = 2h^2 \cdot \frac{du}{d\phi} \cdot \frac{d^2u}{d\phi^2} - 6h^2\xi u^2 \frac{du}{d\phi} + 2h^2u \cdot \frac{du}{d\phi}$$

So, $h^2 \cdot \frac{du}{d\phi}\left[\frac{d^2u}{d\phi^2} + u - \frac{\xi}{h^2} - 3\xi u^2\right] = 0$

Thus, the path of light ray in the interior black hole region is given by $\frac{du}{d\phi} = 0 \dots (9)$

$$\text{Or } \frac{d^2u}{d\phi^2} + u = \frac{\xi}{h^2} + 3\xi u^2 \dots (10)$$

In (10) there is an additional term $3\xi u^2$ to $\frac{d^2u}{d\phi^2} + u = \frac{\xi}{h^2}$ that represents relativistic perturbation. This perturbation indicates a precession phenomenon. Precession of the Perihelion of Mercury was the only experimental evidence against Newtonian Dynamics when Einstein formulated general relativity. Similar equation for geodesics are obtained for Schwarzschild exterior metric.

3. Conclusion

These equations obtained in the interior region of a black hole are similar to the equation of geodesic of a particle obtained in Schwarzschild exterior spacetime; the only difference is that the roles of radial and temporal coordinates have been interchanged and this effect has come up with the interior black hole solution only. Hence, in the interior region of a black hole the geodesic of a particle is the same as that is in the exterior region of an ordinary star which is described by Schwarzschild exterior metric. We note that the interior structure of realistic black holes have not been satisfactorily determined, and is still open to considerable debate.

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