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COMMON FIXED POINT OF TWO CONTINUOUS MAPPINGS ON A COMPLETE METRIC SPACE

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Abstract. The following theorem namely,
"Let (X, d) be a complete metric space and $T, S : X \to X$ be continuous mappings such that

(1) $$d(Sx, Ty) \leq \lambda d(x, y);$$

for all $x, y \in X(x \neq y)$ and for some $\lambda \in [0, 1)$, then S and T have unique common fixed point."
has been proved.

1. Introduction

The classical Banach contraction principle or Fixed point theorem which serves as a bridge between Algebra and Analysis finds an extensible discussion in [1]

2. Result

The following common fixed point theorem has been proved for two continuous mappings on a complete metric space.

Theorem. *Let (X, d) be a complete metric space and $T, S : X \to X$ be continuous mappings such that*

(2) $$d(Sx, Ty) \leq \lambda d(x, y);$$

for all $x, y \in X(x \neq y)$ and for some $\lambda \in [0, 1)$, then S and T have unique common fixed point.

Proof: Let x_0 be an arbitrary point in X, and define

$$x_{2n} = Sx_{2n-1}, n = 1, 2, ...$$

$$x_{2n+1} = Tx_{2n}, n = 0, 1, 2, ...$$

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Suppose that $n = 2m$ for some integer m. Then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_{2m}, x_{2m+1}) \\ &\leq \lambda d(x_{2m-1}, x_{2m}) \end{aligned}$$

Continuing in this way one gets

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda d(x_{2m-1}, x_{2m}) \\ &\cdot \\ &\cdot \\ &\cdot \\ &\leq \lambda^n d(x_0, x_1) \end{aligned} \tag{3}$$

By similar way, one can see that inequality (3) is also true if n is an odd integer. Since $0 \leq \lambda < 1$, the sequence $\{x_n\}$ is a Cauchy sequence and therefore, by completeness of X one finds $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since $\{x_{2n}\} and \{x_{2n+1}\}$ are subsequences of $\{x_n\}$, one gets $\lim_{n\to\infty} x_{2n} = x$ and $\lim_{n\to\infty} x_{2n+1} = x$. Next, since S and T are continuous, one arrives at

$$\begin{aligned} Sx &= S \lim_{n\to\infty} x_{2n-1} \\ &= \lim_{n\to\infty} Sx_{2n-1} \\ &= \lim_{n\to\infty} x_{2n} \\ &= x; \end{aligned} \tag{4}$$

and

$$\begin{aligned} Tx &= T \lim_{n\to\infty} x_{2n} \\ &= \lim_{n\to\infty} Tx_{2n} \\ &= \lim_{n\to\infty} x_{2n+1} \\ &= x \end{aligned} \tag{5}$$

Hence, x is a common fixed point of S and T.

Now, to prove the uniqueness of the common fixed point, suppose that there are $x_1, x_2 \in X$ such that $Sx_1 = x_1 = Tx_1$ and $Sx_2 = x_2 = Tx_2$.Then, by (2),one is led to

$$\begin{aligned} d(x_1, x_2) &= d(Sx_1, Tx_2) \\ &\leq \lambda d(x_1, x_2) \end{aligned}$$

As $0 \leq \lambda < 1$, one finds that $x_1 = x_2$.

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References

[1] Khamsi M. A. and Kirk W. A., *An Introduction to Metric Spaces and Fixed Point Theory*, John Wiley & Sons, New York.

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