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ON $n-$QUASINORMAL OPERATORS

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ABSTRACT. In this paper, bounded operator T on a Hilbert space with the property T^*T^n commutes with T is introduced. Sum and product of such operators have been discussed. It is also discussed that an operator satisfies this property under certain conditions on its real and imaginary parts.

1. INTRODUCTION

We denote $\mathcal{L}(H)$, the space of all bounded linear operators from a complex Hilbert space H to H. For $T \in \mathcal{L}(H)$, the symbols $R(T)$, $N(T)$ and T^* the range, the kernel and the adjoint of T respectively are used. An operator $T \in \mathcal{L}(H)$ is called normal if $T^*T = TT^*$, quasinormal if $T(T^*T) = (T^*T)T$, hyponormal if $T^*T \geq TT^*$.

An operator T is called $n-$normal if T^n is a normal operator. This is equivalent to $T^*T^n = T^nT^*$. These operators have been studied in [5]. $n-$power quasinormal operators have been studied in [3]. T is called $n-$power quasinormal if $(T^*T)T^n = T^n(T^*T)$.

In this paper, we introduce and study a class of $n-$quasinormal operators. We say that $T \in \mathcal{L}(H)$ is $n-$quasinormal if $(T^*T^n)T = T(T^*T^n)$. For $n-$quasinormal operators $S,\ T \in \mathcal{L}(H)$ with $ST = TS = T^*S = ST^* = 0$, we show that $S + T$ is $n-$ quasinormal. It is proved that the product of $n-$quasinormal operators satisfying some commuting conditions is $n-$ quasinormal. We also discuss some conditions on an $n-$quasinormal operator implying normality or hyponormality. For self adjoint operators A and B in $\mathcal{L}(H)$, it is discussed that $T = A + iB$ is $n-$quasinormal under certain commuting conditions on A and B. Some counter examples are also given at appropriate places.

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2. $n-$QUASINORMAL OPERATORS

It is well known that a quasinormal operator on a finite dimensional space is normal. Following example shows that $n-$power quasinormal operators on $\mathbb{C}^3$ need not be normal. It also shows that $n-$power quasinormal operator need not be $n-$quasinormal.

Example 2.1. Let $T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Then $T^3 = T$. Thus $T^{2n} = T^2$.

Now $T(T^*T^{2n}) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ and $(T^*T^{2n})T = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

So $T(T^*T^{2n}) \neq (T^*T^{2n})T$. Thus T is not $2n$-quasinormal.

But $T^{2n}(T^*T) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = (T^*T)T^{2n}$. Thus T is $2n$-power quasinormal.

Sum of two $n-$power quasinormal operators is discussed in [3]. Following is a similar result to that.

Theorem 2.2. *If T and S in $\mathcal{L}(H)$ are $n-$quasinormal and $TS = 0 = ST = T^*S = ST^*$, then $T + S$ is $n-$quasinormal.*

Proof. $((T + S)^*(T + S)^n)(T + S)$

$= (T^* + S^*)(T^n + \binom{n}{1}T^{n-1}S + \binom{n}{2}T^{n-2}S^2 + \ldots\ldots + \binom{n}{n-1}TS^{n-1} + S^n)(T + S)$

$= (T^*T^n + S^*S^n)(T + S) = T^*T^{n+1} + S^*S^{n+1}$. Now

$(T + S)((T + S)^*(T + S)^n)$

$= (T + S)(T^* + S^*)(T^n + \binom{n}{1}T^{n-1}S + \binom{n}{2}T^{n-2}S^2 + \ldots\ldots + \binom{n}{n-1}TS^{n-1} + S^n)$

$= (T + S)(T^*T^n + S^*S^n) = TT^*T^n + SS^*S^n$.

Since T and S are $n-$quasinormal operators we have

$((T+S)^*(T+S)^n)(T+S) = (T+S)((T+S)^*(T+S)^n)$. Hence $T+S$ is $n-$quasinormal. □

The following discusses the conditions for product of two $n-$quasinormal operators to be $n-$quasinormal.

Theorem 2.3. *If T and S in $\mathcal{L}(H)$ are $n-$quasinormal and $TS = ST$, $T^*S = ST^*$, then TS is $n-$quasinormal.*

Proof. $((TS)^*(TS)^n)(TS) = (S^*T^*)T^{n+1}S^{n+1} = S^*S^{n+1}T^*T^{n+1}$. Since S and T are $n-$quasinormal, $(TS)((TS)^*(TS)^n) = TSS^*S^nT^*T^n = S^*S^{n+1}T^*T^{n+1} = ((TS)^*(TS)^n)(TS)$. Hence TS is a $n-$quasinormal operator. □

Corollary 2.4. If T in $\mathcal{L}(H)$ is $n-$quasinormal and S is normal with $TS = ST$, then TS is $n-$quasinormal.

Theorem 2.5. *If T in $\mathcal{L}(H)$ is $n-$quasinormal, then $N(T^n) = N(T^k)$ for each $k \geq n$.*

Proof. Since T is $n-$quasinormal, $(T^*T^n)T = T(T^*T^n)$. Suppose $T^{n+1}(x) = 0$, then $(T^*T^n)Tx = 0$. Thus $T(T^*T^n)x = 0$. So $< T(T^*T^n)x, y >= 0$, for each $y \in H$. Thus $< (T^*T^n)x, T^*y >= 0$, for each $y \in H$. By taking $y = T^nx$ we have, $||(T^*T^n)x|| = 0$. Thus $T^*T^n(x) = 0$. By the similar arguments we see that $T^nx = 0$. Thus $N(T^n) = N(T^{n+1})$. Now result follows as T is $k-$quasinormal for each $k \geq n$. □

Theorem 2.6. *If T in $\mathcal{L}(H)$ is $n-$quasinormal and S is unitary equivalent to T, then S is $n-$quasinormal.*

Proof. Since S is unitary equivalent to T, $S = UTU^*$ for some unitary operator U. Thus $(S^*S^n)S = U(T^*T^n)TU^*$ and $S(S^*S^n) = UT(T^*T^n)U^*$. Since T is $n-$quasi normal, $(S^*S^n)S = S(S^*S^n)$. Hence S is $n-$quasinormal. □

Let $T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then T is 2-quasinormal but $S = XTX^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ is not 2-quasinormal. This shows that $n-$quasinormality is not preserved under similarity.

Theorem 2.7. *If T in $\mathcal{L}(H)$ is $n-$quasinormal and $n-$normal, then T^* is $n-$quasi normal.*

Proof. $((T^*)^*(T^*)^n)T^* = (T(T^nT^*))^* = (T(T^*T^n))^*$
$= ((T^*T^n)T)^* = ((T^nT^*)T)^* = T^*((T^*)^*(T^n)^*)$. Hence T^* is $n-$quasinormal. □

Theorem 2.8. *If T in $\mathcal{L}(H)$ is $n-$quasinormal and $n-$normal, then T is k-normal for each $k \geq n$.*

Proof. Since T is $n-$quasinormal $(T^*T^n)T = T(T^*T^n)$. So $T^*T^{n+1} = TT^*T^n$. Since T is $n-$normal, $T^*T^{n+1} = TT^nT^*$. Thus T is $(n+1)-$normal. Hence by [5], T is $k-$normal for each $k \geq n$. □

It is well known that a normal operator is quasinormal. However $2-$normal operator need not be $2-$quasinormal.

Example 2.9. Let $T = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$, then $T^*T^2 = \begin{pmatrix} 8 & 0 \\ 4 & -8 \end{pmatrix} = T^2T^*$. Hence T is 2-normal. Now $(T^*T^2)T = \begin{pmatrix} 16 & 8 \\ 8 & 20 \end{pmatrix}$ and $T(T^*T^2) = \begin{pmatrix} 20 & -8 \\ -8 & 16 \end{pmatrix}$. Thus $(T^*T^2)T \neq T(T^*T^2)$. Hence T is not 2-quasinormal.

Following is a natural binomial expansion of $(A+B)^n$ for $n \geq 5$.

Lemma 2.10. *Let $A,\ B \in \mathcal{L}(H)$ be such that $A^kB = BA^k$, $B^kA = AB^k$ for $k = 2,\ 3$ and $(AB)^2 = (BA)^2$. Then $(A+B)^n = \sum\limits_{k=0}^{n} \binom{n}{k} A^{n-k}B^k$ for $n \geq 5$.*

Proof. Since $A^kB = BA^k$, $B^kA = AB^k$ for $k = 2,\ 3$ and $(AB)^2 = (BA)^2$, we get
$(A+B)^3 = A^3 + 2A^2B + 2AB^2 + ABA + BAB + B^3$ and
$(A+B)^4 = A^4 + 4A^3B + 4A^2B^2 + 4AB^3 + 2(AB)^2 + B^4$. Hence
$(A+B)^5 = A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5$. Since A and B commute with each term in expansion of $(A+B)^5$. Thus $(A+B)^n = \sum\limits_{k=0}^{n} \binom{n}{k} A^{n-k}B^k$ for $n \geq 5$. $\square$

Now we discuss $n-$quasinormality of an operator under some commuting conditions on its real and imaginary parts.

Theorem 2.11. *Let $T = A + iB$ in $\mathcal{L}(H)$ be such that A and B are selfadjoint. If $B^kA = AB^k$, $A^kB = BA^k$, $k = 2,\ 3$ and $(AB)^2 = (BA)^2$, then T is $n-$quasinormal for each n.*

Proof. $(T^*T^2) = A^3 + iABA + BAB + iB^3$.

$$
\begin{aligned}
\text{Thus } (T^*T^2)T &= A^4 + iABA^2 + (BA)^2 + iB^3A + iA^3B - (AB)^2 + iBAB^2 - B^4 \\
&= A^4 + 2iA^3B + 2iAB^3 - B^4
\end{aligned}
$$

$$
\begin{aligned}
\text{and } T(T^*T^2) &= A^4 + iA^2BA + (AB)^2 + iAB^3 + iBA^3 - (BA)^2 + iB^2AB - B^4 \\
&= A^4 + 2iA^3B + 2iAB^3 - B^4.
\end{aligned}
$$

Therefore $(T^*T^2)T = T(T^*T^2)$. Hence T is $2-$quasinormal.

$T^*T^3 = A^4 + 2iA^3B - 2A^2B^2 + iA^2BA - (AB)^2 - iAB^3 - iBA^3 + 2BA^2B + 2iBAB^2 + (BA)^2 + iB^2AB - B^4 = A^4 + 2iA^3B + 2iAB^3 - B^4$.

$$
\begin{aligned}
\text{Thus } (T^*T^3)T &= A^5 + 2iA^3BA + 2iAB^3A - B^4A + iA^4B - 2A^3B^2 - 2AB^4 - iB^5 \\
&= A^5 + 3iA^4B - 2A^3B^2 + 2iA^2B^3 - 3AB^4 - iB^5
\end{aligned}
$$

and $T(T^*T^3) = A^5 + 2iA^4B + 2iA^2B^3 - AB^4 + iBA^4 - 2BA^3B - 2BAB^3 - iB^5$

$= A^5 + 3iA^4B - 2A^3B^2 + 2iA^2B^3 - 3AB^4 - iB^5.$

Therefore $(T^*T^3)T = T(T^*T^3)$. Hence T is $3-$quasinormal.

$T^*T^4 = A^5 + 3iA^4B - 2A^3B^2 - 2iA^2B^3 + 5AB^4 - iB^5.$

Thus $T(T^*T^4) = A^6 + 3iA^5B - 2A^4B^2 - 2iA^3B^3 + 5A^2B^4 - iAB^5 + iBA^5 - 3BA^4B - 2iBA^3B^2 + 2BA^2B^3 + 5iBAB^4 + B^6$

$= A^6 + 4iA^5B - 5A^4B^2 - 4iA^3B^3 + 7A^2B^4 + 4iAB^5 + B^6.$

and $(T^*T^4)T = A^6 + 3iA^4BA - 2A^3B^2A - 2iA^2B^3A + 5AB^4A - iB^5A + iA^5B - 3A^4B^2 - 2iA^3B^3 + 2A^2B^4 + 5iAB^5 + B^6$

$= A^6 + 4iA^5B - 5A^4B^2 - 4iA^3B^3 + 7A^2B^4 + 4iAB^5 + B^6.$

Therefore $(T^*T^4)T = T(T^*T^4)$. Hence T is $4-$quasinormal. Now we prove that T is $n-$quasinormal for each $n \geq 5$. By Corollary 2.10, $(A+iB)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k}(iB)^k$.

Thus $(A+iB)^*(A+iB)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n+1-k}(iB)^k - \sum_{k=0}^{n} \binom{n}{k}(iB)A^{n-k}(iB)^k$

$= \binom{n}{0} A^{n+1} + \sum_{k=0}^{n-1}[\binom{n}{k+1} - \binom{n}{k}]A^{n-k}(iB)^{k+1} - \binom{n}{n}(iB)^{n+1}.$

$(A+iB)(A+iB)^*(A+iB)^n = \binom{n}{0} A^{n+2} + \sum_{k=0}^{n-1}[\binom{n}{k+1} - \binom{n}{k}]A^{n+1-k}(iB)^{k+1} - \binom{n}{n}A(iB)^{n+1} + \binom{n}{0}A^{n+1}(iB) + \sum_{k=0}^{n-1}[\binom{n}{k+1} - \binom{n}{k}]A^{n-k}(iB)^{k+2} - \binom{n}{n}(iB)^{n+2}$

$= \binom{n}{0} A^{n+2} + [\binom{n}{1} - \binom{n}{0}]A^{n+1}(iB) + \sum_{k=1}^{n-1}[\binom{n}{k+1} - \binom{n}{k}]A^{n+1-k}(iB)^{k+1} - \binom{n}{n}A(iB)^{n+1} + \binom{n}{0}A^{n+1}(iB) + \sum_{k=1}^{n-1}[\binom{n}{k} - \binom{n}{k-1}]A^{n-k+1}(iB)^{k+1} + [\binom{n}{n} - \binom{n}{n-1}]A(iB)^{n+1} - \binom{n}{n}(iB)^{n+2}$

$= \binom{n}{0} A^{n+2} + \binom{n}{1}A^{n+1}(iB) + \sum_{k=1}^{n-1}[\binom{n}{k+1} - \binom{n}{k-1}]A^{n+1-k}(iB)^{k+1} - \binom{n}{n-1}A(iB)^{n+1} - \binom{n}{n}(iB)^{n+2}.$

Similarly $(A+iB)(A+iB)^*(A+iB)^n = \binom{n}{0} A^{n+2} + \binom{n}{1}A^{n+1}(iB) + \sum_{k=1}^{n-1}[\binom{n}{k+1} - \binom{n}{k-1}]A^{n+1-k}(iB)^{k+1} - \binom{n}{n-1}A(iB)^{n+1} - \binom{n}{n}(iB)^{n+2}.$

□

3. Conditions implying normality

We discuss results pertaining to an $n-$quasinormal operator to be a normal operator.

Theorem 3.1. *If T in $\mathcal{L}(H)$ is $n-$quasinormal and $N(T^*) \subset N(T)$, then T is normal.*

Proof. Since T is $n-$quasinormal, $(T^*T - TT^*)T^n = 0$. Thus $(T^*T - TT^*)T^{n-1}y = 0$ for each $y \in R(T)$. Let $x \in R(T)^{\perp} = N(T^*)$. Then $x \in N(T)$. Hence $(T^*T - TT^*)T^{n-1}x = 0$. Thus $(T^*T - TT^*)T^{n-1} = 0$. Hence T is $(n-1)$-quasinormal. Continuing this way, we get T is normal. □

Corollary 3.2. If T in $\mathcal{L}(H)$ is $n-$quasinormal with $R(T) = H$, then T is normal.

Theorem 3.3. *If T in $\mathcal{L}(H)$ is quasinormal and $T + \lambda$ is $n-$quasinormal for some real $\lambda \neq 0$, $n \geq 1$, then T is normal.*

Proof. Since $T + \lambda$ is $n-$quasinormal,
$((T + \lambda)^*(T + \lambda)^n)(T + \lambda) = (T + \lambda)((T + \lambda)^*(T + \lambda)^n)$.
Thus $T^*(T^n + n\lambda T^{n-1} + \binom{n}{2}\lambda^2 T^{n-2} + \ldots + \lambda^n)(T + \lambda) + \lambda(T^n + n\lambda T^{n-1} + \binom{n}{2}\lambda^2 T^{n-2} + \ldots + \lambda^n)(T + \lambda) = (T + \lambda)T^*(T^n + n\lambda T^{n-1} + \binom{n}{2}\lambda^2 T^{n-2} + \ldots + \lambda^n) + \lambda(T + \lambda)(T^n + n\lambda T^{n-1} + \binom{n}{2}\lambda^2 T^{n-2} + \ldots\ldots + \lambda^n)$. Which implies that $T^*(T^n + n\lambda T^{n-1} + \binom{n}{2}\lambda^2 T^{n-2} + \ldots + \lambda^n)T = TT^*(T^n + n\lambda T^{n-1} + \binom{n}{2}\lambda^2 T^{n-2} + \ldots + \lambda^n)$.
So $T^*T^{n+1} + n\lambda T^*T^n + \binom{n}{2}\lambda^2 T^*T^{n-1} + \ldots + \lambda^n T^*T = TT^*T^n + n\lambda TT^*T^{n-1} + \binom{n}{2}\lambda^2 TT^*T^{n-2} + \ldots\ldots + \lambda^n TT^*$. Now using the quasinormality of T, we have $T^*T = TT^*$. Thus T is normal. □

Every quasinormal operator is hyponormal. Under some condition, an $n-$quasinormal operator is hyponormal.

Theorem 3.4. *If T in $\mathcal{L}(H)$ is an $n-$quasinormal operator and $R(T) = R(T^n)$, then T is hyponormal.*

Proof. Let $x = u + v \in H = N(T^*) \oplus \overline{R(T)}$. Suppose $v_n \in R(T)$ be such that $v = \lim\limits_{n\to\infty} v_n$. Let $D = T^*T - TT^*$. Since T is $n-$quasinormal, $DT^n = 0$. $\langle Du,\ u\rangle = \langle T^*T(u),\ u\rangle = ||Tu||^2$. For $y \in H$, $\langle y,\ Dv\rangle = \lim\limits_{n\to\infty}\langle y,\ Dv_n\rangle = 0$. Thus $Dv = 0$. Therefore $\langle Dx,\ x\rangle = \langle Du,\ u\rangle = ||Tu||^2 \geq 0$ for each $x \in H$. Hence T is hyponormal. □

Corollary 3.5. If T in $\mathcal{L}(H)$ is an $n-$quasinormal operator on a finite dimensional Hilbert space with $R(T) = R(T^n)$, then T is normal.

Consider $T : l^2 \to l^2$ defined by $Te_i = e_{i+1}$, if $i = 2k$, $Te_i = 0$, if $i = 2k + 1$. Then T is 2-quasinormal. But $\langle (TT^* - T^*T)e_2,\ e_2\rangle = \langle -T^*e_3,\ e_2\rangle = \langle -e_2,\ e_2\rangle = -1$. Thus T is not hyponormal. This shows that the condition in the Theorem 3.4 is necessary.

Theorem 3.6. *If T in $\mathcal{L}(H)$ is quasinormal and $2-$normal, then T is normal.*

Proof. $(T^*T - TT^*)^2 = (T^*T)^2 - T^*T^2T^* - TT^{*2}T + (TT^*)^2$
$= (T^*T)^2 - T^2T^{*2} - T^{*2}T^2 + TT^*TT^* = T^{*2}T^2 - T^2T^{*2} - T^{*2}T^2 + T^*TTT^*$
$= T^{*2}T^2 - T^2T^{*2} - T^{*2}T^2 + T^2T^{*2} = 0$.
Hence $T^*T - TT^* = 0$. □

Theorem 3.7. *If T in $\mathcal{L}(H)$ is $n-$quasinormal for odd n and $2-$normal, then T is $(n-1)-$quasinormal.*

Proof. Since T is $n-$quasinormal and $n-$normal,
$T^{*n-1}(T^*T - TT^*)^2T^{n-1}$
$= T^{*n-1}(T^*T)^2T^{n-1} - T^{*n-1}T^*T^2T^*T^{n-1} - T^{*n-1}TT^{*2}TT^{n-1} + T^{*n-1}(TT^*)^2T^{n-1}$
$= T^{*n}TT^*T^n - T^{*n}T^2T^*T^{n-1} - T^{*n-1}TT^{*2}T^n + T^{*n-1}TT^*TT^*T^{n-1}$
$= T^{*n+1}T^{n+1} - T^{*n}T^2T^*T^{n-1} - T^{*n-1}TT^{*2}T^n + T^{*n-1}TT^*TT^*T^{n-1}$
$= T^{*n+1}T^{n+1} - T^{*n+1}T^{n+1} - T^{*n+1}T^{n+1} + T^{*n+1}T^{n+1} = 0$
Thus $T^{*n-1}(T^*T - TT^*)^2T^{n-1} = 0$. Hence $(T^*T - TT^*)T^{n-1} = 0$. Thus T is $(n-1)-$quasinormal. □

Theorem 3.8. *If T in $\mathcal{L}(H)$ is $n-$quasinormal for even n and $2-$normal with $(T^*T)^2 = T^{*2}T^2$, then T is $(n-1)-$quasinormal.*

Proof. Since T is $n-$quasinormal and $2-$normal with $(T^*T)^2 = T^{*2}T^2$,
$T^{*n-1}(T^*T - TT^*)^2T^{n-1}$
$= T^{*n-1}(T^*T)^2T^{n-1} - T^{*n-1}T^*T^2T^*T^{n-1} - T^{*n-1}TT^{*2}TT^{n-1} + T^{*n-1}(TT^*)^2T^{n-1}$
$= T^{*n}TT^*T^n - T^{*n}T^2T^*T^{n-1} - T^{*n-1}TT^{*2}T^n + T^{*n-1}TT^*TT^*T^{n-1}$
$= T^{*n+1}T^{n+1} - T^{*n}T^2T^*T^{n-1} - T^{*n-1}TT^{*2}T^n + T^{*n-2}(T^*T)^2T^*T^{n-1}$
$= T^{*n+1}T^{n+1} - T^{*n+1}T^{n+1} - T^{*n+1}T^{n+1} + T^{*n-2}T^{*2}T^2T^*T^{n-1} = 0.$
Thus $T^{*n-1}(T^*T - TT^*)^2T^{n-1} = 0$. Hence $(T^*T - TT^*)T^{n-1} = 0$. Thus T is $(n-1)-$quasinormal.

□

Following corollary immediately follows from above theorems.

Corollary 3.9. If T in $\mathcal{L}(H)$ is $n-$quasinormal and $2-$normal with $(T^*T)^2 = T^{*2}T^2$, then T is normal.

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