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The trustees of P C Vaidya Sanman Nidhi Trust feel honored and are proud in making an arrangement with the Department of Mathematics, Gujarat University; so that each lecture delivered under "Prof. I. H. Sheth Mathematics Lecture Series" at Mathematics Department (Gujarat University) from the donation by Dr. Vasudha Kulshreshtha, will be published in the forthcoming issue. The current article is the first lecture under the series which was delivered on 4th September, 2015. It is hoped that this gesture of the trust will be welcome by the Mathematics community associated with Mathematics Today and shall take due advantage.

Conditions Implying Normality in Operator Theory and Prof. I. H. Sheth

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The present paper is based on Prof. I. H. Sheth lecture delivered by author on September 04, 2015 at the Department of Mathematics, Gujarat University, Ahmedabad. With a deep sense of respect to Prof Sheth, the lecture and subsequent paper are being presented.

Prof. U. N. Singh, Prof. I. H. Sheth and Prof. B. S. Yadav pioneered the development of research in Operator Theory in Gujarat. Prof Sheth joined for Ph D under Prof. Singh at M. S. University of Baroda. It so happened that Prof. Singh decided to develop Operator Theory at that time, and he trained I. H. Sheth and B. S. Yadav in Operator Theory leading to their Ph. D.s. Dr. Sheth moved to Gujarat University and started developing Operator Theory in Ahmedabad; where as Dr. Yadav joined Sardar Patel University and initiated Operator Theory at Vallabh Vidyanagar in collaboration with Prof. P. B. Ramanujan and S. M. Patel. Prof Sheth successfully guided 7 Ph. D. students in Operator Theory. He concentrated his study on non normal Hilbert space operator. In this paper, I will describe the contribution of Prof. Sheth on a particular theme viz conditions implying normality for Hilbert space operators. The contribution of Prof. Sheth in Mathematical Education deserves a separate story.

2. Preliminaries and Basic Definitions

Suppose C is the field of complex numbers. A vector space X over C is called an normed linear space if there is a function $\| . \| : X \to [0, \infty)$ (called norm on X) such that $\|x\| =$

0 iff $x = 0$, $||x + y|| \le ||x|| + ||y||$ and $||\alpha x|| = |\alpha|\ ||x||$, $\alpha \in C$ and $x, y \in X$. If X is a normed linear space the one can define a metric d on X induced by the norm on X by $d(x, y) = ||x - y||$, $x, y \in X$. A vector space X over C is called an Inner Product Space if there is a function $<.\ ,.> : X\times X \to C$ (known as inner product on X) such that $<x, x> \ge 0$ for all $x \in X$ and for $x \in X$, $<x, x> = 0$ iff $x = 0$; $<x+z, y> = <x, y> + <z, y>$; $<\alpha x, y> = \alpha<x, y>$ and $<x, y> = \overline{<y, x>}$, $x, y, z \in X$, $\alpha \in C$. On an inner product space X, define $||x|| = \sqrt{<x, x>}$, $x \in X$. Then $(X, ||.||)$ is a normed linear space. Thus on X, we have geometrical as well as topological structure on X. A complete normed linear space is called a Banach space and a complete inner product space is called a Hilbert space.

Now onwards we assume that H is a Hilbert space. A linear operator $T : H \to H$ is called bounded if there is $\beta > 0$ such the $||Tx|| \le \beta\ ||x||$ for all $x \in X$. (which is equivalent to T being continuous on H). The space BL(H) of all bounded linear maps from H to H is a Banach space with the norm $||T|| = \sup\{||Tx|| : x \in X \text{ and } ||x|| \le 1\}$. In fact BL(H) is a Banach algebra.

For $T \in BL(H)$, the set $\sigma(T) = \{ \lambda \in C: T - \lambda I$ does not have inverse in $BL(H)\}$ is called the spectrum of T. It is non empty compact subset of C. The set $\sigma p(T)$ of all eigenvalues of T is called the point spectrum of T; the set $W(T) = \{ <Tx, x> : x \in H, ||x|| = 1\}$ is known as the numerical range of T. It is convex and bounded by $||T||$. Also $\sigma(T) \subset \overline{W(T)}$. For $T \in BL(H)$, $r(T) = \sup\{ |\lambda| : \lambda \in \sigma(T)\}$ and $w(T) = \sup\{ |\lambda| : \lambda \in W(T)\}$ are known as the spectral radius and numerical radius of T respectively.

For $T \in BL(H)$, one can associate another operator $T^* \in BL(H)$ (known as the adjoint of T) such that $<Tx, y> = <x, T^*y>$, $x, y \in H$. (it is analogous to the conjugate of a complex number, fortunately T and T* do not commute). T is called self-adjoint of $T^* = T$. In general T and T* do not commute. The operator $T \in BL(H)$ is called a normal operator, if $T^*T = TT^*$ (equivalently $||Tx|| = ||T^*x||$ for all $x \in H$). $T \in BL(H)$ is called a hyponormal operator, if $T^*T \le TT^*$ (equivalently $||Tx|| \le ||T^*x||$ for all $x \in H$). If $\Omega \subset C$ is compact and μ is a measure on Ω, for $\varphi \in L\infty(\Omega, \mu)$, define $(M\varphi f)(t) = \varphi(t) f(t)$, $t \in \Omega$. Then $M\varphi : L2(\Omega, \mu) \to L2(\Omega, \mu)$ is a bounded linear operator on the Hilbert space $L2(\Omega, \mu)$. It is a normal operator. Normal operators behave like complex function in some sense. In fact if $T \in BL(H)$ is a normal operator, then H is iosmetrically isomorphic to the Hilbert space $L2(\Omega, \mu)$ and T is unitarily equivalent to $M\varphi$ in $BL(L2(\Omega, \mu))$ for some $\varphi \in L\infty(\Omega, \mu)$. (This is the well known spectral theorem for normal operators; there are other versions of the spectral theorem too). Normal operators are well studied and they have nice properties. Many authors have tried to define and studied some weak versions of normal operators.

T is called quasi-normal if (T*T) commutes with T. T is called subnormal if T is a restriction of some normal operator on closed invariant subspace. (i.e N is a normal operator on a Hilbert space K and H is a closed subspace of K with $N(H) \subset H$ and $N|H = T$.) (quasinormal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal)

3. Partial Isometry and Normality

An operator T is called isometry if $\|Tx\| = \|x\|$ for all $x \in H$. (equivalently $T^*T = I$) and T is called partial isometry if T^*T is a projection. ($P \in BL(H)$ is called projection if P2 = P and P is self adjoint.) T is called power partial isometry if Tk is a partial isometry for some $k \in N$. An onto isometry is called unitary operator (equivalently $T^*T = TT^* = I$). For $T \in BL(H)$, the set $M(T) = \{x \in H : \|Tx\| = \|T\| \|x\|\}$ is called the maximal vector subspace of T.

An operator T is a non-zero partial isometry iff $\|T\| = 1$ and $\overline{R(T)} \subset M(T^*)$ (or $\overline{R(T^*)} \subset M(T)$. T is called normloid if $r(T) = \|T\|$ (equivalently $w(T) = \|T\|$). The following results were obtained by N C Shah and I H Sheth [4]. If T is noramloid isometry and $M(T) \subset M(Tk)$ for some k, then T is power partial isometry. If $R(T^*) \subset M(T)$ and $N(T) \subset N(T^*)$, then T is hyponormal and it can be expressed as the direct sum of a scalar multiple of isometry and 0. If T is contraction with Tk = T for some positive integer $k \geq 2$, Then T is a normal partial isometry. If T is contraction with Tk = I for some positive integer $k \geq 2$, Then T is unitary. If T is partial isometry and $M(T) = M(T^*)$, then T is normal. T is called quasi-hyponormal if $T^*(T^*T - TT^*)T \geq 0$. (hyponormal $\Rightarrow$ quasi-hyponormal). If $\overline{R(T)} \subset M(T)$, then T is quasi-hyponormal and either $0 \in W(T)$ or T is a scalar multiple of unitary. Hence if $\overline{R(T)} \subset M(T)$, and 0 is an extreme point of W(T), then T is a scalar multiple of normal partial isometry. If T is isometry with T*k is hyponormal for some k, then T is unitary.

4. Centroid Operators and Normality

$T \in BL(H)$ is called normaloid if $w(T) = \|T\|$ (equivalently $r(T) = \|T\|$). $T \in BL(H)$ is called convexoid if $\text{conv}(\sigma(T)) = \overline{W(T)}$. $T \in BL(H)$ is called centroid of $T - zTI$ is normaloid, where zT is the centre of smallest circular disk dT containing the spectrum σ(T) of T. If P is any property, we say that T is restriction P if T restricted to every invariant subspace of T has property P. (e.g. T is called restriction convexoid if restriction of T on each closed invariant subspace of H is convexoid.) We say that T is reduction P if T restricted to every reducing subspace of T has property P.

The paper [1] of Nakamoto and Sheth contains the following: If T is centroid and 0 ∉ dT, then T−1 is centroid. If T is Restriction centroid, then isolated point of the spectrum of T is an eigenvalue of T. (This is a property of a normal operator.) If T is Restriction centroid with finite spectrum, then T is normal. If T is Restriction centroid such that σ(T) has only one limit point, then T is normal. An operator T is called compact if every bounded sequence {xn} in H, {Txn} has a convergent subsequence. A compact Restriction centroid operator is normal.

5. Compactness and Normaity

An operator T on H is called isoloid if every isolated point of σ(T) is an eigenvalue of T. The result described in this paragraph were obtained by Prasanna and Sheth [2]. Let T ∈ BL(H) with compact imaginary part such that all invariant subspaces of T are reducing. Then T is reduction isoloid. A compact reduction convexoid operator is normal. Also a reduction convexoid operator with compact imaginary part is normal. If a reduction convexoid operator T such that αT + βT* is compact (where α ≠ 0 or β ≠ 0 |α| ≠ |β|.), then T is normal. T is called polynomially compact if p(T) is compact for some polynomial p(x) ∈ C[x]. If T is reduction G1 w. r. t. the spectrum such that all its finite dimensional eigenspaces of T are reducing under T and αT + βT* is polynomially compact (where α ≠ 0 or β ≠ 0 |α| ≠ |β|.), then T is normal. T is called paranormal if ||Tx||2 ≤ ||T2x|| ||x|| for all x ∈ H. (Equivalently T*2T − 2λT*T +λ2 ≥ 0.) (quasi-hyponormal⇒ paranormal). If T is paranormal, then T is reduction isoloid. A parnormal operator with finite spectrum is normal. If T is reduction normaloid and αT + βT* is compact (where α ≠ 0 or β ≠ 0 |α| ≠ |β|.), then T is normal. If T is reduction normaloid and W(T/Y) = $\overline{W(T/Y)}$ for every closed reducing subspace Y of T, then T is normal. T ∈ BL(H) is called operator of class P if $\overline{\text{extreme } W(T)}$ ∩ W(T) ⊂ π0(T), the set of all isolated eigenvalues of finite multiplicity. If T is reduction P and Im(T) is compact, then T is normal.

6. Operators of Class M and normality

T ∈ BL(H) is called of class M if $\|T^*x\|^2 \leq \|T^2x\|\,\|x\|$ for each x ∈ H (equivalently $T^{*2}T^2 - 2\lambda (TT^*) + \lambda^2 I \geq 0$ for all $\lambda \geq 0$). The restriction of an operator of class M on a closed invariant subspace is of class M. Quasi-nilpotent operator of class M is 0. Suppose T is of class M. Then (i) If 0 is isolated point of σ(T), then 0 ∈ $\sigma_p(T)$. (ii) M(T*) ⊂ M(T). (iii) M(T) = M(T^n) for each n ≥ 1 (iv) M(T) is invariant under T and (v) T/M(T) is paranormal. Suppose T is of class M. If T^2 is normal, then T is normal. If T^n is uniary for some n, then T is unitary. If T* is hyponormal, then T is normal. These results were obtained by Shah and Sheth [5].

7. Quasi hyponormality and Normality

T ∈ BL(H) is called quasi−hyponormal if $T^{*2}T - (T^*T)^2 \geq 0$. In the paper [3], Sheth obtained the results: If T is quasi−hyponormal, then T is normaloid. Hence a quasi hyponormal quasi nilpotent operator is 0. Invertible quasi−hyponormal operators are hyponormal. A quasi−hyponormal operator with dense range is hyponormal. Restriction of a quasi−hyponormal operator on a reducing subspace is quasi−hyponormal. If T = A + iB, (A, B are self-adjoint) is hyponormal such that B is compact, then T is normal. If T = A + iB, (A, B are self-adjoint) is quasi-hyponormal such that σ(T) ∩ R ⊂ {0} and B is compact, then T is normal [3].

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Sr ક્રમ	Book પુસ્તક	Publication પ્રકાશન
૧	ગણિતજ્ઞ રામાનુજન	ગ્રંથલોક પ્રકાશન
૨	Mathematician Ramanujan	સુગણિતમ ટ્રસ્ટ
૩	સુરેખ બીજગણિત	ગ્રંથ નિર્માણ બોર્ડ
૪	અરૂપ બીજગણિત	ગ્રંથ નિર્માણ બોર્ડ
૫	Abstract Algebra	PHI
૬	શ્રેણીઓ અને શ્રેઢીઓ	ગ્રંથ નિર્માણ બોર્ડ
૭	જાદુઈ ચોરસ	ગુજરાત સાયન્સ એકેડેમી
૮	અવિભાજ્ય સંખ્યાઓ	સીમિત પ્રકાશન
૯	કબૂતર ખાનાનો સિદ્ધાંત	સુગણિતમ ટ્રસ્ટ
૧૦	Prime Numbers	ગુજરાત સાયન્સ એકેડેમી
૧૧	ગમ્મતમય ગણિત	સીમિત પ્રકાશન
૧૨	ભૌમિતિક રચનાઓ	સુગણિતમ ટ્રસ્ટ
૧૩	Translation into Gujarati of the book: Introductory lessons in modern mathematics by P. L. Bhatnagar	University Book Production centre, Gujarat Stale
૧૪	Linear equations	Gujarat State School Text Books Board
૧૫	Linear Programming	University Book Production centre, Gujarat Stale
૧૬	Vectors and their applications in physics.	University Book Production centre, Gujarat Stale