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On Invariance of Approximation and Coapproximation in Metric Linear Spaces

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Abstract

We discuss invariance of approximation, coapproximation and orthogonality in metric linear spaces. It is shown that isometric linear maps preserve approximation properties and orthogonality. The results proved in this paper generalize and extend several known results on the subject.

Keywords: Proximinal set, coproximinal set, Chebyshev set, co-Chebyshev set, quasi Chebyshev set, quasi co-Chebyshev set.

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The notion of invariant best approximation was introduced and discussed by Meinardus [5] in normed linear spaces. Thereafter, Brosowski [1] proved some intersting results on invariance of best approximation in normed linear spaces using fixed point theory. Various generalizations of their results were later obtained by L. Habiniak, L.A. Khan and A.R. Khan, S.P. Singh, A. Smoluk, P.V. Subrahmanyam and others (see [3]). The author extended some of these results to metric spaces in [8]. A kind of approximation called coapproximation (by Papini and Singer [10]), was introduced and discussed in normed linear spaces by Franchetti and Furi [2]. Subsequently, Papini and Singer [10], Geetha S. Rao and her coworkers developed this theory to a considerable extent (see [11]-[14] and references cited therein). Many results on this topic appeared in normed linear spaces, metric linear spaces, metric spaces and other abstract spaces (see e.g. [6],[7],[9]-[14] and references cited therein). Mazaheri and Zadeh [4] discussed certain maps which preserve approximation properties in normed linear spaces.

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In this paper, we prove a result on coapproximation in metric spaces which is analogous to the invariance principle of Meinardus on best approximation. We also discuss invariance of approximation, coapproximation and orthogonality in metric linear spaces. The results proved in this paper generalize and extend various known results (including those of [1],[4] and [5]) on the subject. To start with, we recall a few definitions.

Let X be a nonempty set and T a self map on X. An element $x \in X$ is said to be $T-$**invariant** if $Tx = x$.

Let G be a subset of a metric space (X, d) and $x \in X$. An element $g_0 \in G$ is said to be a **best approximation (best coapproximation)** to x if $d(x, g_0) \leq d(x, g)$ $(d(g_0, g) \leq d(x, g))$ for all $g \in G$. The set of all such $g_0 \in G$ is denoted by $P_G(x)(R_G(x))$.

The set G is said to be **proximinal (coproximinal)** if $P_G(x)(R_G(x))$ is non-empty for all $x \in X$. It is said to be **Chebyshev (co-Chebyshev)** if $P_G(x)(R_G(x))$ is a singleton for each $x \in X$.

A proximinal(coproximinal) subset G of a metric space (X, d) is called **quasi-Chebyshev [4](quasi co-Chebyshev)** if the set $P_G(x)(R_G(x))$ is compact for all $x \in X$.

Suppose (X, d) is a metric linear space and $x, y \in X$, x is said to be **orthogonal** to y, written as $x \perp y$, if

$$d(x, 0) \leq d(x, \alpha y)$$

for every scalar α.

We say that $G_1 \perp G_2$ if $g_1 \perp g_2$ for all $g_1 \in G_1,\ g_2 \in G_2$.

Let G be a subspace of X.

We define **metric complement**, $\hat{G}$ of G as

$$\begin{aligned}\hat{G} &= \{x \in X : x \perp G\} \\ &= \{x \in X : x \perp g \ for\ all\ g \in G\} \\ &= \{x \in X : d(x,0) \leq d(x, \alpha g)\ for\ all\ g \in G\ and\ all\ scalars\ \alpha\}\end{aligned}$$

We define the **cometric complement**, $\check{G}$ of G as

$$\begin{aligned}\check{G} &= \{x \in X : G \perp x\} \\ &= \{x \in X : g \perp x \ for\ all\ g \in G\} \\ &= \{x \in X : d(g,0) \leq d(g, \alpha x)\ for\ all\ g \in G\ and\ all\ scalars\ \alpha\}\end{aligned}$$

A subspace G of X is called **orthogonal complemented**[4] if either G is Chebyshev and $\hat{G}$ is a subspace of X or G is co-Chebyshev and $\check{G}$ is a subspace of X.

We first discuss some basic facts concerning $\hat{G}$ and $\check{G}$

Proposition 1: Let G be a linear subspace of a metric linear space (X, d) and $x \in X$. Then the following are true:

(a) g_0 is a best approximation to $x \in X$ if and only if $(x - g_0) \in \hat{G}$.

(b) If $(x - g_0) \in \check{G}$ then $g_0 \in G$ is a best coapproximation to $x \in X$.

(c) For $g_0 \in G$, $\alpha g_0 \in R_G(\alpha x)$ for every scalar α if and only if $(x - g_0) \in \check{G}$

Proof:(a)

$$\begin{aligned}
(x - g_0) \in \hat{G} &\Leftrightarrow (x - g_0) \perp G \\
&\Leftrightarrow (x - g_0) \perp g \ for\ all\ g \in G \\
&\Leftrightarrow d(x - g_0, 0) \leq d(x - g_0, \alpha g) \ for\ all\ g \in G\ and\ all\ scalars\ \alpha \\
&\Leftrightarrow d(x, g_0) \leq d(x, g_0 + \alpha g) \ for\ all\ g \in G\ and\ all\ scalars\ \alpha \\
&\Leftrightarrow d(x, g_0) \leq d(x, g') \ for\ all\ g' \in G \\
&\Leftrightarrow g_0 \in P_G(x).
\end{aligned}$$

(b)

$$\begin{aligned}
(x - g_0) \in \check{G} &\Rightarrow G \perp (x - g_0) \\
&\Rightarrow g \perp (x - g_0) \ for\ all\ g \in G \\
&\Rightarrow d(g, 0) \leq d(g, \alpha(x - g_0)) \ for\ all\ g \in G\ and\ all\ scalars\ \alpha \\
&\Rightarrow d(g, 0) \leq d(g, \alpha x - \alpha g_0) \ for\ all\ g \in G\ and\ all\ scalars\ \alpha \\
&\Rightarrow d(g + \alpha g_0, \alpha g_0) \leq d(g + \alpha g_0, \alpha x) \ for\ all\ g \in G\ and\ all\ \alpha \\
&\Rightarrow d(g' - \alpha g_0, 0) \leq d(g', \alpha x) \ for\ all\ g \in G\ and\ all\ scalars\ \alpha \\
&\Rightarrow d(g', \alpha g_0) \leq d(g', \alpha x) \ for\ all\ g' \in G\ and\ all\ scalars\ \alpha \\
&\Rightarrow d(g', g_0) \leq d(g', x) \ for\ all\ g' \in G \\
&\Rightarrow g_0 \in R_G(x).
\end{aligned}$$

(c) Let $\alpha g_0 \in R_G(\alpha x)$ for all scalars α i.e. $d(\alpha g_0, g) \leq d(\alpha x, g)\ for\ all\ g \in G$ and all scalars α. This gives $d(\alpha x - \alpha g_0, g - \alpha g_0) \geq d(g - \alpha g_0, 0)$ for all $g \in G$ and all scalars α i.e. $d(g', \alpha(x - g_0)) \geq d(g', 0)$ for all $g' \in G$ and all scalars α. Therefore $g' \perp (x - g_0)$ for all $g' \in G$ i.e. $G \perp (x - g_0)$ and so $(x - g_0) \in \check{G}$.

Conversly, let $(x-g_0)\in\check{G}$. Then, $G\perp(x-g_0)$ i.e. $g\perp(x-g_0)$ for all $g\in G$. This implies that $d(g,\alpha(x-g_0))\geq d(g,0)$ for all $g\in G$ and all scalars α i.e. $d(g,\alpha x-\alpha g_0)\geq d(g,0)$ for all $g\in G$ and all scalars α. Therefore, $d(g+\alpha g_0,\alpha x)\geq d(g,0)$ for all $g\in G$ and all scalars α i.e. $d(g',\alpha x)\geq d(g'-\alpha g_0,0)$ for all $g'\in G$ and all scalars α i.e. $d(g',\alpha g_0)\leq d(g',\alpha x)$ for all $g'\in G$ and all scalars α. Therefore $\alpha g_0\in R_G(\alpha x)$ for all scalars α.

Notes:**(i)** $x\in\hat{G}\Leftrightarrow x-0\in\hat{G}\Leftrightarrow 0\in P_G(x)\Leftrightarrow d(x,0)=d(x,G)\Leftrightarrow x\in P_G^{-1}(0). Therefore\ \hat{G}=P_G^{-1}(0)$.

(ii) $g_0\in P_G(x)\Leftrightarrow(x-g_0)\in\hat{G}\Leftrightarrow(x-g_0)-0\in\hat{G}\Leftrightarrow 0\in P_G(x-g_0)$.

G. Meinardus [5] (see also [1]) proved the following result on invariance of best approximation:

(**Invariance Principle**). Let T be a linear mapping from a normed linear space X onto itself such that $\|Tf\|=\|f\|$ for each $f\in X$. Moreover, assume that $T(M)=M$. Then the element Tu_0 is a best approximation to Tf from M, whenever u_0 is a best approximation to f.

If f is invariant, i.e., if $Tf=f$, and $u_0\in P_M(f)$, then also $Tu_0\in P_M(f)$.

If f is invariant and its best approximation u_0 is unique, then u_0 is invariant under T.

If f is invariant and no invariant element from M can be best approximation , then f has either no best approximation or at least two.

Analogous to the Invariance Principle of Meinardus on best approximation (see also [8]), we have the following result on invariance of best coapproximation in metric spaces:

Theorem 1: Let T be an isometry on a metric space (X,d) i.e. $d(Tx,Ty)=d(x,y)$ for all $x,y\in X$, and G be a subset of X such that $T(G)=G$. Then

(i) $T(R_G(x))\subseteq R_G(Tx)$.

(ii) If x is $T-$ invariant then $T(R_G(x))\subseteq R_G(x)$.

(iii) If x is $T-$ invariant and if $R_G(x)=\{g_0\}$ is a singleton then $Tg_0=g_0$.

(iv) If x is $T-$ invariant and if $\{g\in G:Tg=g\}\bigcap R_G(x)=\phi$ then either $R_G(x)=\phi$ or $R_G(x)$ has more than one point.

Proof: **(i)**. Let $Tg_0\in T(R_G(x))$ i.e. $g_0\in R_G(x)$. Let $g\in G$ be arbitrary. Then $T(G)=G$ implies the existence of $u\in G$ such that $g=Tu$. Consider $d(Tg_0,g)=d(Tg_0,Tu)=d(g_0,u)\leq d(x,u)=d(Tx,Tu)=d(Tx,g)\ for\ all\ g\in G$. This implies that $Tg_0\in R_G(Tx)$ whenever g_0 is a best coapproximation to x i.e. Tg_0 is a best coapproximation to Tx whenever g_0 is a best coapproximation to x.

(ii) Suppose $g_0\in R_G(x)$. Then (i)$\Rightarrow Tg_0\in R_G(Tx)$ i.e. $Tg_0\in R_G(x)$ i.e. $T(R_G(x))\subseteq R_G(x)$.

(iii). By (ii), $Tg_0\in\{g_0\}\ i.e.\ Tg_0=g_0$.

(iv). By (ii), $g_0, Tg_0 \in R_G(x)$. But by the hypothesis, no invariant element can be best coapproximation, therefore $Tg_0 \neq g_0$. So, if $Tg_0 = g_0$ then best coapproximation to x does not exist in G i.e. $R_G(x) = \phi$. If $Tg_0 \neq g_0$ then x has at least two best coapproximations.

Without the assumption $T(G) = G$ in the above theorem, we have **Proposition 2**: Let (X, d) be a metric space. If $T : X \to X$ is an isometry then for all subspaces G of $X$and all $x \in X$,

$$T(P_G(x)) = P_{T(G)}(Tx) \text{ and } T(R_G(x)) = R_{T(G)}(Tx).$$

Proof: Since T is an isometry, $d(Tx, Ty) = d(x, y)$ *for all* $x, y \in X$. The proofs follow from

$$d(x, g_0) \leq d(x, g) \text{ for all } g \in G \Leftrightarrow d(Tx, Tg_0) \leq d(Tx, Tg) \text{ for all } Tg \in T(G)$$

and

$$d(g_0, g) \leq d(x, g) \text{ for all } g \in G \Leftrightarrow d(Tg_0, Tg) \leq d(Tx, Tg) \text{ for all } Tg \in T(G).$$

Suppose (X, d) and (Y, d') are metric spaces. A map $T : X \to Y$ is called **approximation preserving (coapproximation preserving)** if for all subspaces G of X and all $x \in X$,

$$T(P_G(x)) = P_{T(G)}(Tx) \ (T(R_G(x)) = R_{T(G)}(Tx))$$

Proposition 2 shows that if (X, d) is a metric space then every isometry $T : X \to X$ is approximation preserving (coapproximation preserving).

The following theorem (proved in [4] for normed linear spaces) shows that isometric linear maps preserve approximation properties and orthogonality:

Theorem 2: Suppose (X, d) and (Y, d') are metric linear spaces and $T : X \to Y$ is an onto linear map which is an isometry. Then

(a) A subspace G of X is proximinal (coproximinal) in X if and only if $T(G)$ is proximinal (coproximinal) in Y.

(b) A subspace G of X is Chebyshev (co-Chebyshev) in X if and only if $T(G)$ is Chebyshev (co-Chebyshev) in Y.

(c) For all $x, y \in X,\ x \perp y \Leftrightarrow Tx \perp Ty$ i.e. T preserves orthogonality.

(d) For a subspace G of X, $T(\hat{G}) = T(\hat{G})(T(\check{G}) = T(\check{G}))$.

(e) If G is a subspace of X, then G is orthogonal complemented in X if and only if $T(G)$ is orthogonal complemented in Y.

(f) If G is a subspace of X and T is continuous, then G is quasi Chebyshev (quasi co-Chebyshev) if and only if $T(G)$ is quasi Chebyshev (quasi co-Chebyshev).

Proof: **(a)** G is proximinal in $X \Leftrightarrow P_G(x) \neq \phi$ *for all* $x \in X \Leftrightarrow T(P_G(x)) \neq \phi \Leftrightarrow P_{T(G)}(Tx) \neq \phi$ *by Proposition* $2 \Leftrightarrow T(G)$ *is proximinal in* $T(X) = Y$.

G is coproximinal in $X \Leftrightarrow T(G)$ is coproximinal in $T(X) = Y$ also follows from Proposition 2.

(b) Since T is an isometry, T preserves approximation (coapproximation) and so the result follows.

(c)$x \perp y \Leftrightarrow d(x,0) \leq d(x,\alpha y)$ *for all scalars* $\alpha \Leftrightarrow d(Tx,T0) \leq d(Tx,T(\alpha y))$ for all scalars $\alpha \Leftrightarrow d(Tx,0) \leq d(Tx,\alpha Ty)$ *for all scalars* $\alpha \Leftrightarrow Tx \perp Ty.$

(d) First we show that $T(\hat{G}) = \hat{T(G)}$

$$\hat{G} = \{x \in X : x \perp G\}, \hat{T(G)} = \{y \in Y : y \perp T(G)\}.$$

Let $y \in T(\hat{G})$. Then $y = Tx,\ x \in \hat{G}$. Now, $x \in \hat{G} \Rightarrow x \perp G \Rightarrow Tx \perp T(G) \Rightarrow y \perp T(G) \Rightarrow y \in \hat{T(G)}$. Therefore, $T(\hat{G}) \subseteq \hat{T(G)}$.

Suppose $y \in \hat{T(G)}$ *i.e.* $y \perp T(G)$. Since T is onto, $y = Tx,\ x \in X$ and so $Tx \perp T(G)$ i.e. $Tx \perp Tg$ *for all* $g \in G$ and therefore

$$d(Tx,0) \leq d(Tx,\alpha Tg) \text{ for all } g \in G \text{ and all scalars } \alpha$$

i.e. $d(Tx,0) \leq d(Tx,T(\alpha g))$ *for all* $g \in G$ *and all scalars* α. Therefore, $d(x,0) \leq d(x,\alpha g)$ *for all* $g \in G$ *and all scalars* α *i.e.* $x \perp g$ *for all* $g \in G$ *i.e.* $x \perp G$ *and so* $x \in \hat{G}$. Therefore $y = Tx \in T(\hat{G})$ and so $\hat{T(G)} \subseteq T(\hat{G})$. Hence $T(\hat{G}) = \hat{T(G)}$.

Now, we show that $T(\check{G}) = \check{T(G)}$.

Let $y \in T(\check{G})$ *i.e.* $y = Tx,\ x \in \check{G}$. Now, $x \in \check{G} \Rightarrow G \perp x \Rightarrow g \perp x$ *for all* $g \in G \Rightarrow d(g,0) \leq d(g,\alpha x)$ *for all* $g \in G$ *and all scalars* $\alpha \Rightarrow d(Tg,0) \leq d(Tg,\alpha Tx)$ *for all* $g \in G$ *and all scalars* $\alpha \Rightarrow Tg \perp Tx$ *for all* $g \in G \Rightarrow Tg \perp y$ *for all* $g \in G \Rightarrow y \in \check{T(G)}$. So, $T(\check{G}) \subseteq \check{T(G)}$

Suppose $y \in T(\check{G})$. So, $T(G) \perp y$ *i.e.* $T(G) \perp Tx,\ x \in X$ *i.e.* $Tg \perp Tx$ *for all* $g \in G$. This implies that $d(Tg,0) \leq d(Tg,\alpha Tx) = d(Tg,T(\alpha x))$ for all $g \in G$ and all scalars α. Therefore $d(g,0) \leq d(g,\alpha x)$ *for all* $g \in G$ and all scalars α *i.e.* $G \perp x$ *and so* $x \in \check{G}$. This implies $y \in T(\check{G})$. Therefore, $T(\check{G}) \subseteq \check{T(G)}$ and hence $T(\check{G}) = \check{T(G)}$.

(e) G is orthogonal complemented in $X \Leftrightarrow$ either G is Chebyshev and $\hat{G}$ is a subspace of X, or G is co-Chebyshev and $\check{G}$ is a subspace of $X \Leftrightarrow$ either $T(G)$ is Chebyshev and $T(\hat{G}) = \hat{T(G)}$ is a subspace of Y, or $T(G)$ is co-Chebyshev and $T(\check{G}) = \check{T(G)}$ is a subspace of Y.

The result now follows from (b), (d), linearity and ontoness of T.

(f) Suppose G is quasi Chebyshev in X. To show $T(G)$ is quasi Chebyshev in Y. Let $z \in Y$ and

$\{u_n\}$ a sequence in $P_{T(G)}(z)$. Since T is onto, $z = T(x),\ x \in X$. Therefore, $\{u_n\}$ is a sequence in $P_{T(G)}(Tx) = T(P_G(x))$ and so $u_n = Tv_n,\ v_n \in P_G(x)$. Since G is quasi Chebyshev, there exist a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $\{v_{n_i}\} \to v_0 \in P_G(x) \Rightarrow \{Tv_{n_i}\} \to Tv_0 \in T(P_G(x))$ i.e. $\{u_{n_i}\} \to Tv_0 \in P_{T(G)}(z)$. Hence $P_{T(G)}(z)$ is compact.

Converse part follows by using the properties of T^{-1}.

The quasi co-Chebyshevity part can be proved on the same lines.

Note: If T preserves approximation and $x \perp y$ then $Tx \perp Ty$ i.e. T preserves orthogonality. Since $x \perp y, d(x, 0) \leq d(x, < y >)$ *i.e.* $0 \in P_{<y>}(x)$. Therefore $0 = T0 \in P_{T<y>}(Tx)$ *i.e.* $0 \in P_{<Ty>}(Tx)$ *as* T *is linear i.e.* $d(Tx, 0) \leq d(Tx, < Ty >)$ *i.e.* $Tx \perp Ty$.

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