Mathematics Today Vol.32 (June & December 2016) 84-92
ISSN 0976-3228

DOUBLE SUMMATION FORMULAE

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Abstract

Using classical Beta function of first kind, two double summation formulae are developed for functions of two variables involving Logarithm and square root of reciprocal functions. These in turn are used to derive several identities involving infinite series in Pochhamer symbol and Gamma functions. Also, the main result is proved by using Pochhamer symbol and series manipulation techniques. Their special cases also studied as applications through combinatorial identities.

2010 MSC Subject classification: $05A10, 33B15, 33B99, 33E20$

Keywords: Beta function, Gamma function, Pochhamer symbol.

1 Introduction

The Gamma function is defined as [1]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad Re(z) > 0, \tag{1}$$

and if z is nonintegral, satisfies

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{2}$$

The Beta function defined as ([1], [3])

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad Re(p) > 0, \quad Re(q) > 0. \tag{3}$$

This is related to the Gamma function by [1]

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad Re(p) > 0, \quad Re(q) > 0.$$

The factorial function is defined by [1]

$$(p)_n = \prod_{k=1}^{n}(p+k-1) = p(p+1)(p+2)\cdots(p+n-1), \qquad n \geq 1, \tag{4}$$

and for $n = 0$, $(p)_0 = 1,\ p \neq 0$. Obviously, $(1)_n = n!$. Also,

$$(p)_n = \frac{\Gamma(p+n)}{\Gamma(p)}, \tag{5}$$

where p is neither zero nor a negative integer.

The series

$$(1-z)^{-p} = \sum_{k=0}^{\infty} \frac{(p)_k}{k!} z^k, \quad |z| < 1, \tag{6}$$

is well known; and the series manipulative techniques are given by [1]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k), \tag{7}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k). \tag{8}$$

2 Main Result

Theorem: If $|x| < 1, |y| < 1$ with $x \neq 0, y \neq 0$ and $p > 0, q > 0$, then

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(p)_k (q)_n}{(p+q)_{k+n}} x^k y^n = \frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty} \frac{(p)_k}{(p+q)_k} x^{k+1} + \sum_{n=0}^{\infty} \frac{(q)_n}{(p+q)_n} y^{n+1}\right]. \tag{9}$$

Proof. Using (3), consider

$$B(p+k, q+n) = \int_0^1 t^{p+k-1}(1-t)^{q+n-1}dt.$$

Using relation between Beta and Gamma function, we can write

$$\frac{\Gamma(p+k)\Gamma(q+n)}{\Gamma(p+q+k+n)} = \int_0^1 t^{p+k-1}(1-t)^{q+n-1}dt.$$

Multiplying both sides by $x^k y^n$, where $|x| < 1$, $|y| < 1$ with $x \neq 0$, $y \neq 0$, gives

$$\begin{aligned}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^k y^n &= \int_0^1 t^{p+k-1}(1-t)^{q+n-1}x^k y^n dt \\ &= \int_0^1 t^{p-1}(1-t)^{q-1}(xt)^k[y(1-t)]^n dt.\end{aligned}$$

Taking double summation from 0 to ∞, we get

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^k y^n = \int_0^1 t^{p-1}(1-t)^{q-1}\sum_{k=0}^{\infty}(xt)^k\sum_{n=0}^{\infty}\{y(1-t)\}^n dt.$$

Both infinite series on right hand side are geometric series hence employing partial fractions, above equation can be written in the form

$$\begin{aligned}&\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^k y^n \\ &= \int_0^1 \frac{t^{p-1}(1-t)^{q-1}}{(1-xt)\{1-y(1-t)\}}dt. \\ &= \frac{1}{x+y-xy}\int_0^1\left(\frac{x}{1-xt}+\frac{y}{1-y(1-t)}\right)t^{p-1}(1-t)^{q-1}dt \\ &= \frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty}x^{k+1}\int_0^1 t^{p+k-1}(1-t)^{q-1}dt + \sum_{n=0}^{\infty}y^{n+1}\int_0^1 t^{p-1}(1-t)^{q+n-1}dt\right].\end{aligned}$$

Observed that, the integrals inside infinite series are beta functions, we can write

$$\begin{aligned}r.h.s. &= \frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty}B(p+k,q)x^{k+1} + \sum_{n=0}^{\infty}B(p,q+n)y^{n+1}\right] \\ &= \frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty}\frac{\Gamma(p+k)\Gamma(q)}{\Gamma(p+q+k)}x^{k+1} + \sum_{n=0}^{\infty}\frac{\Gamma(p)\Gamma(q+n)}{\Gamma(p+q+n)}y^{n+1}\right].\end{aligned}$$

After some simplification, we finally arrive at

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^k y^n = \frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty}\frac{(p)_k}{(p+q)_k}x^{k+1} + \sum_{n=0}^{\infty}\frac{(q)_n}{(p+q)_n}y^{n+1}\right].$$

□

3 Two double series formulae

Corollary: If $|x| < 1, |y| < 1$ with $x \neq 0$, $y \neq 0$ then

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{k!n!}{(k+n+1)!}x^k y^n = -\frac{log(1-x)log(1-y)}{x+y-xy}, \tag{10}$$

and

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{2})_n}{(k+n)!}x^k y^n = \frac{x(1-x)^{-\frac{1}{2}} + y(1-y)^{-\frac{1}{2}}}{x+y-xy}. \tag{11}$$

Proof. For $p = q = 1$, (9) reduces to

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_k(1)_n}{(2)_{k+n}}x^ky^n &= \frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty}\frac{(1)_k}{(2)_k}x^{k+1}+\sum_{n=0}^{\infty}\frac{(1)_n}{(2)_n}y^{n+1}\right]\\ &= \frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}+\sum_{n=0}^{\infty}\frac{y^{n+1}}{n+1}\right]\\ &= \frac{1}{x+y-xy}\left[\sum_{k=1}^{\infty}\frac{x^{k}}{k}+\sum_{n=1}^{\infty}\frac{y^{n}}{n}\right]\\ &= \frac{-log(1-x)-log(1-y)}{x+y-xy}.\end{aligned}$$

Here, using factorial notation, we get

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{k!n!}{(k+n+1)!}x^ky^n = -\frac{log(1-x)log(1-y)}{x+y-xy}.$$

Next, for $p = q = \frac{1}{2}$, (9) reduces to

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{2})_n}{(1)_{k+n}}x^ky^n &= \frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{(1)_k}x^{k+1}+\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{(1)_n}y^{n+1}\right]\\ &= \frac{1}{x+y-xy}\left[x\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{k!}x^{k}+y\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!}y^{n}\right].\end{aligned}$$

Hence,

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{2})_n}{(k+n)!}x^ky^n = \frac{x(1-x)^{-\frac{1}{2}}+y(1-y)^{-\frac{1}{2}}}{x+y-xy}.$$

This is (11).

Alternate proof of (11)**:**

Taking $q = 1-p$, in (9), leads us to

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(1-p)_n}{(p+1-p)_{k+n}}x^ky^n\\ =\ &\frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty}\frac{(p)_k}{(p+1-p)_k}x^{k+1}+\sum_{n=0}^{\infty}\frac{(1-p)_n}{(p+1-p)_n}y^{n+1}\right].\end{aligned}$$

This further reduces to

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(1-p)_n}{(k+n)!}x^ky^n &= \frac{1}{x+y-xy}\left[x\sum_{k=0}^{\infty}\frac{(p)_k}{k!}x^{k}+y\sum_{n=0}^{\infty}\frac{(1-p)_n}{n!}y^{n}\right]\\ &= \frac{1}{x+y-xy}\left[x(1-x)^{-p}+y(1-y)^{-1+p}\right].\end{aligned}\tag{12}$$

Taking $p = \frac{1}{2}$ in (12), gives (11).

□

4 Applications

Theorem of section 2 is used to derive the following identities.

$$\frac{1}{(p+q)_n}\sum_{k=0}^{n}(p)_k(q)_{n-k} = \sum_{k=0}^{n}2^{-n+k-1}\frac{(p)_k+(q)_k}{(p+q)_k}, \tag{13}$$

$$\frac{1}{(n+1)}\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}^{-1} = \sum_{k=0}^{n}\frac{2^{-n+k}}{k+1}, \tag{14}$$

$$\frac{1}{n!}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{n-k} = 2^{-n}\sum_{k=0}^{n}\frac{\Gamma(2k+1)}{2^k[\Gamma(k+1)]^2}, \tag{15}$$

$$\sum_{k=0}^{n}(-1)^{n-k}(p)_k(q)_{n-k} = \frac{(p)_{n+1}+(-1)^n(q)_{n+1}}{p+q+n}, \tag{16}$$

$$\sum_{k=0}^{2n}(-1)^k(p)_k(p)_{2n-k} = \frac{(p)_{2n+1}}{p+n}, \tag{17}$$

$$\sum_{k=0}^{2n}(-1)^k\begin{pmatrix} 2n \\ k \end{pmatrix}^{-1} = \frac{2n+1}{n+1}, \tag{18}$$

$$\sum_{k=0}^{2n}(-1)^k\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{2n-k} = \frac{2\left(\frac{1}{2}\right)_{2n+1}}{2n+1}, \tag{19}$$

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{4}+k)\Gamma(\frac{3}{4}+n)}{(k+n)!}x^ky^n = \frac{\sqrt{2}B(\frac{1}{2},\frac{1}{2})}{x+y-xy}\left[x(1-x)^{-\frac{1}{4}}+y(1-y)^{-\frac{3}{4}}\right], \tag{20}$$

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{3}{4}+k)\Gamma(\frac{1}{4}+n)}{(k+n)!}x^ky^n = \frac{\sqrt{2}B(\frac{1}{2},\frac{1}{2})}{x+y-xy}\left[x(1-x)^{-\frac{3}{4}}+y(1-y)^{-\frac{1}{4}}\right]. \tag{21}$$

Proof. (of (13)): Taking $y = x$ in (9), we get

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^{k+n} = \frac{1}{2x-x^2}\left[\sum_{k=0}^{\infty}\frac{(p)_k}{(p+q)_k}x^{k+1}+\sum_{n=0}^{\infty}\frac{(q)_n}{(p+q)_n}x^{n+1}\right].$$

Observed that, in square bracket on right hand side, both terms are infinite series which can be written as

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^{k+n} = \frac{1}{2(1-\frac{x}{2})}\sum_{k=0}^{\infty}\frac{(p)_k+(q)_k}{(p+q)_k}x^k.$$

In this using series manipulative techniques, gives

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(p)_k(q)_{n-k}}{(p+q)_n}x^n = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}\sum_{k=0}^{\infty}\frac{(p)_k+(q)_k}{(p+q)_k}x^k.$$

This can be put in the form

$$\sum_{n=0}^{\infty}\frac{x^n}{(p+q)_n}\sum_{k=0}^{n}(p)_k(q)_{n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{2^{n+1}}\frac{(p)_k+(q)_k}{(p+q)_k}x^{n+k}.$$

Again, using series manipulative techniques, we further get

$$\sum_{n=0}^{\infty} \frac{x^n}{(p+q)_n} \sum_{k=0}^{n} (p)_k (q)_{n-k} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{2^{n-k+1}} \frac{(p)_k + (q)_k}{(p+q)_k} x^n.$$

By comparing the coefficients of x^n on both sides, we get (13). □

Proof. (of (14)): Taking $q = p$ in equation (13), we get

$$\frac{1}{(2p)_n} \sum_{k=0}^{n} (p)_k (p)_{n-k} = \sum_{k=0}^{n} 2^{-n+k} \frac{(p)_k}{(2p)_k}. \tag{22}$$

With $p = 1$, (22) paticularizes to

$$\frac{1}{(2)_n} \sum_{k=0}^{n} (1)_k (1)_{n-k} = \sum_{k=0}^{n} 2^{-n+k} \frac{(1)_k}{(2)_k},$$

that is,

$$\frac{1}{(n+1)!} \sum_{k=0}^{n} k!(n-k)! = \sum_{k=0}^{n} 2^{-n+k} \frac{k!}{(k+1)!},$$

or

$$\frac{1}{(n+1)} \sum_{k=0}^{n} \frac{k!(n-k)!}{n!} = \sum_{k=0}^{n} 2^{-n+k} \frac{1}{k+1}.$$

This proves (14). □

Proof. (of (15)): Putting $p = \frac{1}{2}$ in equation (22) yields,

$$\frac{1}{(1)_n} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{n-k} = \sum_{k=0}^{n} 2^{-n+k} \frac{\frac{1}{2}\ \frac{3}{2}\ \frac{5}{2}\ \cdots \frac{2k-1}{2}}{(1)_k}$$

that is,

$$\frac{1}{n!} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{n-k} = 2^{-n} \sum_{k=0}^{n} \frac{1.2.3\ldots(2k-1)}{k!}$$

This when written in the form of Gamma function, leads us to (15). □

Proof. (of (16)): With $y = -x$, equation (9) reduces to

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(p)_k (q)_n}{(p+q)_{k+n}} x^k (-x)^n = \frac{1}{x^2} \left[\sum_{k=0}^{\infty} \frac{(p)_k}{(p+q)_k} x^{k+1} + \sum_{n=0}^{\infty} \frac{(q)_n}{(p+q)_n} (-x)^{n+1} \right].$$

Here the expression on the r.h.s. can be written as

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(p)_k (q)_n}{(p+q)_{k+n}} x^{k+n} = \sum_{n=1}^{\infty} \frac{(p)_n + (-1)^{n+1} (q)_n}{(p+q)_n} x^{n-1},$$

which in view of series manipulation techniques, takes the form

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} \frac{(p)_k (q)_{n-k}}{(p+q)_n} x^n = \sum_{n=0}^{\infty} \frac{(p)_{n+1} + (-1)^n (q)_{n+1}}{(p+q)_{n+1}} x^n.$$

Thus (16) follows by comparing the coefficients of x^n on both sides. □

Proof. (of (17)): For $q = p$, (16) gives

$$\sum_{k=0}^{n}(-1)^{n-k}(p)_k(p)_{n-k} = \frac{(p)_{n+1} + (-1)^n(p)_{n+1}}{2p+n}.$$

Replacing n by $2n$, this becomes

$$\sum_{k=0}^{2n}(-1)^{2n-k}(p)_k(p)_{2n-k} = \frac{(p)_{2n+1} + (-1)^{2n}(p)_{2n+1}}{2p+2n},$$

that is,

$$\sum_{k=0}^{2n}(-1)^{k}(p)_k(p)_{2n-k} = \frac{(p)_{2n+1}}{p+n}.$$

□

Proof. (of (18)): For $p = 1$, equation (17) leads us to

$$\sum_{k=0}^{2n}(-1)^{k}(1)_k(1)_{2n-k} = \frac{(1)_{2n+1}}{1+n} = \frac{(2n+1)!}{1+n},$$

or

$$\sum_{k=0}^{2n}(-1)^{k}\frac{k!(2n-k)!}{(2n)!} = \frac{2n+1}{n+1}.$$

Alternatively,

$$\sum_{k=0}^{2n}(-1)^{k}\begin{pmatrix}2n\\k\end{pmatrix}^{-1} = \frac{2n+1}{n+1}.$$

□

Proof. (of (19)): For $p = \frac{1}{2}$, equation (17) gives

$$\sum_{k=0}^{2n}(-1)^{k}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{2n-k} = \frac{\left(\frac{1}{2}\right)_{2n+1}}{\frac{1}{2}+n} = \frac{2\left(\frac{1}{2}\right)_{2n+1}}{2n+1}.$$

□

Proof. (of 20): Taking $p = \frac{1}{4}$ in (12) , we get

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_k(\frac{3}{4})_n}{(k+n)!}x^ky^n = \frac{1}{x+y-xy}\left[x(1-x)^{-\frac{1}{4}} + y(1-y)^{-\frac{3}{4}}\right]. \tag{23}$$

From equation (5), we have

$$\left(\frac{1}{4}\right)_k\left(\frac{3}{4}\right)_n = \frac{\Gamma(\frac{1}{4}+k)\Gamma(\frac{3}{4}+n)}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})} = \frac{\Gamma(\frac{1}{4}+k)\Gamma(\frac{3}{4}+n)}{\pi\sqrt{2}}.$$

which puts (23) in the form:

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{4}+k)\Gamma(\frac{3}{4}+n)}{(k+n)!}x^ky^n &= \frac{\pi\sqrt{2}}{x+y-xy}\left[x(1-x)^{-\frac{1}{4}} + y(1-y)^{-\frac{3}{4}}\right]\\ &= \frac{\sqrt{2}B(\frac{1}{2},\frac{1}{2})}{x+y-xy}\left[x(1-x)^{-\frac{1}{4}} + y(1-y)^{-\frac{3}{4}}\right].\end{aligned}$$

□

Proof. (of 21): Here (21) follows at once by taking $p = \frac{3}{4}$ in (12) and applying same techniques of proof of identity (20). □

By using suitable values of z in (2), few interesting results are mentioned below. $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2},\ \Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right) = 2\pi,\ \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2}{\sqrt{3}}\pi,\ \Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{7}{8}\right) = \frac{2}{\sqrt{2-\sqrt{2}}}\pi,$
$\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{11}{12}\right) = \frac{4}{\sqrt{6-\sqrt{2}}}\pi,\ \Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{6}{12}\right) = \frac{4}{\sqrt{6+\sqrt{2}}}\pi.$

5 Alternate proof of main result

In this section we present an alternate proof of main result (9). We define the notation

$$\delta(k,n) = \frac{(p)_k (q)_k}{(p+q)_{k+n}}. \tag{24}$$

Consider,

$$\begin{aligned}
&(x+y-xy)\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\delta(k,n)x^k y^n \\
= \ & x\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\delta(k,n)x^k y^n + y\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\delta(k,n)x^k y^n - xy\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\delta(k,n)x^k y^n \\
= \ & \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\delta(k,n)x^{k+1} y^n + \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\delta(k,n)x^k y^{n+1} - \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\delta(k,n)x^{k+1} y^{n+1}.
\end{aligned}$$

On replacing k by $k-1$ and n by $n-1$, this reduces to

$$\begin{aligned}
&(x+y-xy)\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\delta(k,n)x^k y^n \\
= \ & \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\delta(k-1,n)x^k y^n + \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\delta(k,n-1)x^k y^n - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\delta(k-1,n-1)x^k y^n \\
= \ & \sum_{k=1}^{\infty}\left[\delta(k-1,0)x^k + \sum_{n=1}^{\infty}\delta(k-1,n)x^k y^n\right] \\
& + \sum_{n=1}^{\infty}\left[\delta(0,n-1)y^n + \sum_{k=1}^{\infty}\delta(k,n-1)x^k y^n\right] - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\delta(k-1,n-1)x^k y^n.
\end{aligned}$$

On rearranging the terms, we get

$$\begin{aligned}
(x+y-xy)\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\delta(k,n)x^k y^n = & \sum_{k=1}^{\infty}\delta(k-1,0)x^k + \sum_{n=1}^{\infty}\delta(0,n-1)y^n \\
& + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}[\delta(k-1,n) + \delta(k,n-1) - \delta(k-1,n-1)]x^k y^n.
\end{aligned} \tag{25}$$

Now, simplification of square brackets on the right hand sides gives

$$\begin{aligned}
& \delta(k-1,n)+\delta(k,n-1)-\delta(k-1,n-1) \\
= & \frac{(p)_{k-1}(q)_n}{(p+q)_{k+n-1}}+\frac{(p)_k(q)_{n-1}}{(p+q)_{k+n-1}}-\frac{(p)_{k-1}(q)_{n-1}}{(p+q)_{k+n-2}} \\
= & \frac{(p)_{k-1}(q)_{n-1}}{(p+q)_{k+n-1}}[(q+n-1)+(p+k-1)-(p+q+k+n-2)] \\
= & 0.
\end{aligned}$$

Thus, (25) becomes

$$(x+y-xy)\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\delta(k,n)x^k y^n=\sum_{k=1}^{\infty}\frac{(p)_{k-1}}{(p+q)_{k-1}}x^k+\sum_{n=1}^{\infty}\frac{(q)_{n-1}}{(p+q)_{n-1}}y^n.$$

Hence,

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^k y^n=\frac{1}{x+y-xy}\left[\sum_{k=1}^{\infty}\frac{(p)_{k-1}}{(p+q)_{k-1}}x^k+\sum_{n=1}^{\infty}\frac{(q)_{n-1}}{(p+q)_{n-1}}y^n\right].$$

This can also be written as

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^k y^n=\frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty}\frac{(p)_k}{(p+q)_k}x^{k+1}+\sum_{n=0}^{\infty}\frac{(q)_n}{(p+q)_n}y^{n+1}\right],$$

when $|x|<1, |y|<1$.

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