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DYNAMICS OF SYMBOLS

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This article is dedicated to Prof. A. M. Vaidya and Prof. I. H. Sheth

Abstract. Spaces that evolve with time are called dynamical systems. Usually such systems are studied by expressing them as solutions of differential equations. But this does not help always as either the system cannot be precisely described by differential equations or the said equations don't have any closed form solutions. So the time here is discretized and these systems are studied as discrete dynamical systems. Many times dynamical systems are studied by discretizing both time and state space. The basic idea lies in taking partition of the state space into finite number of regions, each of which can be labelled with some symbol. Time is then discretized by taking iterates of all points in the space. Each itinerary in the state space then corresponds to an infinite sequence of symbols, where the symbols are those of the region in the partition given by the trajectory of the point. This 'Dynamics of Symbols' though leads to a loss of some information, but is very useful in capturing the essence of any dynamics.

This article does not contain any new ideas, but is just a brief write-up of basics on the subject.

1. INTRODUCTION

The evolution of trajectories of a discrete dynamical system (X, f) is equivalent to the symbolic dynamics in an appropriate symbol system. Any dynamical system (X, f) induces a dynamical system $(\sum, s)$, where $\sum \subseteq A^{\mathbb{Z}}$ or $(A^{\mathbb{N}})$, endowed with the product topology, for some symbol set A and s is the left shift $s(x)_n = x_{n+1}$ with $s(\sum) \subset \sum$. The shift map σ gives a continuous self map on $\sum$. In this article, we shall present the dynamical behaviour of the systems $(\sum, s)$.

The foundation to such a study of symbolic dynamics is credited to Jacques Hadamard, who used infinite symbol sequences in his analysis of geodesic flows on negatively curved surfaces. Hadamard's techniques of using symbols were later adopted and extended by other authors. However, this field became more prominent due to the studies by Marston Morse and Gustav Hedlund, who provided the first systematic study of symbolic dynamical systems as objects of interest in their own right. This study was later used in the mathematical analysis of codes and finite-alphabet communication systems using the techniques of ergodic theory in the pioneering work of C.E. Shannon on the mathematical theory of communication. Today, this area is an independent field of research in its own and has applications ranging from study of partial differential equations to coding theory to ergodic theory itself.

The reader is referred to the excellent book "An Introduction to Symbolic Dynamics and Coding" by Douglas Lind and Brian Marcus, and the references therein, for further study.

1.1. **Motivation and the beginning.** Dynamical systems were first studied by Newton's description of the dynamics in physics. Such a motion is governed by a system of differential equations. A typical study here was the motion observed in solar system, say the two body problem. This led to the development of methods of solving differential equations which became the main subject of mathematical study for a long time. Later more powerful machinery of complex analysis and Fourier analysis were developed to deal with more difficult differential equations.

Now finding explicit equations depicting the motion of systems turned out to be very difficult or almost impossible except for a few elementary cases. This difficulty led to the study using more abstract approach, i.e. to seek answers to the questions concerning the motion of systems without expressing the systems by explicit equations. This led to the birth of the abstract subject of Topology.

With the tools from Topology - many abstract methods and results were formulated, namely the Poincare recurrence theorem and the Ergodic Theorems by Birkhoff and von Neumann.

Those days mathematicians were busy investigating the description of geodesic curves on a Riemannian manifold. Such curves happen to be solutions to certain differential equations for minimizing length of a path joining given points on the manifold and they appear naturally as trajectories of moving particles.

One of such a study was Hadamard's paper concerning geodesics on Riemannian manifolds with negative curvature.

Hadamard proved that there exists a finite set of closed curves $G = \{c_1, c_2, \ldots, c_n\}$, such that each homotopy class of solution curves represents traversing these curves in a particular order. In other words the trajectory is described by a double infinite sequence $a = \ldots a_{-1}, a_0, a_1, \ldots$ ($a_i \in G\ \forall i$). He then proved that every such class contains exactly one geodesic, thus establishing a

$1-1$ correspondence between all bounded geodesics and all double infinite sequence over the alphabet G.

Hadamard's idea was later exploited by Harold Marston Morse. A system which does not contain a proper subsystem is called minimal. All periodic systems are minimal. But at that time an example of a nonperiodic minimal flow was unknown. Morse first proved that Hadamard's coding is continuous (i.e. two sequences are close to each other if and only if the corresponding geodesics are close to each other). And then he proved that periodic sequences correspond to closed geodesics. Using a modern language, Morse showed that the geodesic flow (observed along integer time moments) contains the full symbolic shift $(G^{\mathbb{Z}}, s)$ as a subflow. Finally, along with Thue, he constructed the famous example of a minimal nonperiodic sequence known today as the Thue-Morse sequence. This sequence is obtained inductively by starting with a 0 and then, in consecutive steps, using the substitution $0 \to 01$, $1 \to 10$.

$$0 \to 01 \to 0110 \to \ldots \to 01101001100101101001011001101001\ldots$$

With this contribution Morse was able to get rid of differential equations and use combinatorial tools such as blocks, patterns, codes, and substitutions to describe trajectories.

Gustav Arnold Hedlund then compiled all such results known till then and formulated the area of abstract symbolic systems.

1.2. **Basic topological dynamics.** Let (X, d) be a metric space and let $f : X \to X$ be a continuous function. We study the behaviour of each point x under repeated actions of f and the pair (X, f) is referred as a dynamical system.

A point $x \in X$ is called *periodic* if $f^n(x) = x$ for some positive integer n, where $f^n = f \circ f \circ f \circ \ldots \circ f$ (n times). The least such n is called the *period* of the point x. For $x \in X$, if there exists a $\delta > 0$ such that for each $\epsilon > 0$ there exists $y \in X$ and a positive integer n such that $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta$, then f is said to be *sensitive* at x. If f is sensitive at each point $x \in X$, f has *sensitive dependence on initial conditions* or is simply called *sensitive*. A map f is called expansive(δ-expansive) if for any pair of distinct elements $x, y \in X$, there exists $k \in \mathbb{N}$ such that $d(f^k(x), f^k(y)) > \delta$. A map f is called *transitive* if for any pair of non-empty open sets U, V in X, there exist a positive integer n such that $f^n(U) \bigcap V \neq \phi$. A map f is called *totally transitive* if f^n is transitive for each $n \in \mathbb{N}$. A map f is called *weakly mixing* if for any pairs of non-empty open sets U_1, U_2 and V_1, V_2 in X, there exists $n \in \mathbb{N}$ such that $f^n(U_i) \bigcap V_i \neq \phi$ for $i = 1, 2$. It is known that for any continuous self map f, if f is weakly mixing and $U_1, U_2, \ldots U_n, V_1, V_2, \ldots V_n$ are non-empty open sets, then there exists a $k \geq 1$ such that $f^k(U_i) \bigcap V_i \neq \phi$ for $i = 1, 2, \ldots n$. A map f is called mixing or *topologically mixing* if for each pair of non-empty open sets U, V in X, there exists a positive integer k such that $f^n(U) \bigcap V \neq \phi$ for all $n \geq k$.

We now define the notion of *topological entropy*.

Let (X, d) be a compact metric space and let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ has a finite subcover. Let $\mathcal{L}$ be the collection of all finite subcovers and let $\mathcal{U}^*$ be the subcover with minimum cardinality, say $N_{\mathcal{U}}$. Define $H(\mathcal{U}) = logN_{\mathcal{U}}$. Then $H(\mathcal{U})$ is defined as the *entropy* associated with the open cover $\mathcal{U}$. If $\mathcal{U}$ and $\mathcal{V}$ are two open covers of X, define, $\mathcal{U} \vee \mathcal{V} = \{U \bigcap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. For a self map f on X, $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$ is also an open cover of X. Define,

$$h_{f,\mathcal{U}} = \lim_{n\to\infty} \frac{H(\mathcal{U}\vee f^{-1}(\mathcal{U})\vee f^{-2}(\mathcal{U})\vee\ldots\vee f^{-n+1}(\mathcal{U}))}{n}$$

Then $\sup h_{f,\mathcal{U}}$, where $\mathcal{U}$ runs over all possible open covers of X is known as the *topological entropy of the map f* and is denoted by $h(f)$.

For the system (X, f), let (X_m, f_m) denote $(X \times X \times \ldots \times X, f \times f \times \ldots \times f)$, where the cartesian product is taken m number of times. Then if (X, f) has topological entropy $h(f)$, $h(f_m) = mh(f)$.

2. Dynamics of Symbols

2.1. **Basics.** Consider the unit interval $I = [0, 1)$ and the map f that sends each $x \in I$ to $\{2x\}$, the fractional part of $2x$. We are interested in the orbit $x, f(x), f^2(x) = f(f(x)), \ldots$. However, we will break I into just two parts, $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1)$. We assign to x a symbolic trajectory $x_0x_1x_2\ldots$ where $x_i$is 0 or 1 depending as $f^i(x)$ is in I_0 or I_1. Here we see that the trajectory of x is completely determined by its symbolic trajectory.

What symbolic trajectories can appear in such a scheme? - All binary sequences except those that end in $111\ldots$. This exception can be removed by working instead with closed intervals $I = [0, 1], I_0 = [0, 1/2]$ and $I_1 = [1/2, 1]$, and mapping sequences to points instead of the other way around. Beginning with a binary sequence $x_0x_1x_2\ldots$, we can assign to it the unique point

$$x = \bigcap_{i=1}^{\infty} f^{-i}(I_{x_i})$$

that has that symbolic itinerary. Then, for example, 1/2 will arise from two symbolic trajectories, corresponding to the two binary expansions $.1000\ldots$ and $.0111\ldots$.

Our system now has a convenient symbolic representation. If $x = .x_0x_1x_2\ldots$ then $f(x) = \{2x\} \approx s(x) = .x_1x_2x_3\ldots$. We shift the symbolic sequence to the left and lop off the initial symbol. The key to the utility of symbolic dynamics is that the dynamics is given by a simple coordinate shift. The benefit with this representation is, for instance, we can easily identify the points of period 3 (that is, solutions of $f^3(x) = x$). We look out for solutions of $s^3(x) = x$. They are the eight points with repeating symbolic trajectories $x_0x_1x_2x_0x_1x_2\ldots$.

2.2. **Full shifts and subshifts.** We let $\mathcal{A}$ denote a symbol set or alphabet, which for now we assume to be finite. The (two-sided) full $\mathcal{A}$-shift is the dynamical system consisting of the set of bi-infinite symbol sequences, together with the shift map s that shifts all coordinates to the left. More formally, our space is $\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i \in \mathbb{Z}} : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z}\}$ and the map $s : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{Z}$ satisfies $(sx)_i = x_{i+1}$. If $\mathcal{A} = \{0, 1, \dots, n-1\}$ we call $\mathcal{A}^{\mathbb{Z}}$ the full n-shift.

Bi-infinite sequences are an indication that the shift map is invertible. However, we may also consider the one-sided $\mathcal{A}$-shift ($\mathcal{A}^{\mathbb{N}}$), with the truncating shift map as described above.

A subshift or shift space is a closed subset of some full shift $\mathcal{A}^{\mathbb{Z}}$ that is invariant under the action of s. For example, the set of binary sequences that do not contain the string 11 is a subshift of the 2-shift.

More generally, let $\mathcal{F}$ be any set of finite strings (also called words or blocks) of symbols of $\mathcal{A}$. The set of sequences that do not contain any word of $\mathcal{F}$ is a subshift $X_{\mathcal{F}}$ of $\mathcal{A}^{\mathbb{Z}}$. In fact, an easy topological argument shows that every subshift is of this type only.

If $X_{\mathcal{F}}$ is determined by a finite set $\mathcal{F}$ of forbidden words, we call $X_{\mathcal{F}}$ a subshift of finite type, or SFT for short. This is the most fully studied class of symbolic dynamical systems, and the one that has been exploited most in the analysis of general dynamical systems. The systems originally considered by Hadamard were of this type.

Let M be a $n \times n$ square $\{0, 1\}$ matrix. Let $\sum_M = \{(x_n) \in \mathcal{A}^{\mathbb{Z}} : M_{x_i x_{i+1}} = 1, i \in \mathbb{N}\}$. Then, $\sum_M$ is a closed shift invariant subset of $\mathcal{A}^{\mathbb{Z}}$ and is an example of a subshift of finite type. Conversely, every subshift of finite type can also be represented by a square $\{0, 1\}$ matrix. In such a case, the matrix M is called the transition matrix for the space $\sum_M$.

Let M be a square $\{0, 1\}$ matrix. A square $\{0, 1\}$ matrix is irreducible if for every pair of indices i and j there is an $l > 0$ with $(M^l)_{ij} > 0$. Fix an index i and let $p(i) = \gcd\{l : (M^l)_{ij} > 0\}$. This is called the period of the index i. When M is irreducible, period of every index is same and is called the period of M. If the matrix has period one, it is said to be aperiodic. M is irreducible and aperiodic if and only if there exists $r \in \mathbb{N}$ such that for all $k \geq r$, M^k is strictly positive.

We give some of the known results.

1. Let $\sum$ be a subshift of $\mathcal{A}^{\mathbb{Z}}$. Then, $x \in \sum$ is a point of sensitivity for the system $(\sum, s)$ if and only if x is not isolated.

2. Let M be a square $\{0, 1\}$ transition matrix. Then, $\sum_M$ is transitive if and only if M is irreducible.

3. Let M be a square $\{0, 1\}$ transition matrix. Then, $\sum_M$ is topological mixing if and only if M is irreducible and aperiodic.

4. Let M be a square $\{0, 1\}$ transition matrix. Then, the following are equivalent.

(1) $\sum_M$ is totally transitive

(2) $\sum_M$ is weakly mixing.

(3) $\sum_M$ is topological mixing.

Let (X, f) be embedded into the subshift of finite type given via the transition matrix $M = (m_{ij})_{n\times n}$.

Let $S = \{(i_1, i_2) : 1 \leq i_1, i_2 \leq n\}$. Let M^{*2} be a matrix indexed by entries of S defined as,

$M^{*2}_{(i_1,i_2),(j_1,j_2)} = m_{i_1 j_1} m_{i_2 j_2}$

Then M^{*2} is a square matrix of order $2n$, and can be considered as the tensor product of M with itself.

When X was being decomposed into n parts, the transition matrix M provided the dynamical behaviour of the system giving the details of how any two given components of the system interact. In the matrix generated above, $M^{*2}_{(i_1,i_2),(j_1,j_2)}$ gives the details of simultaneous interaction of the $i_1 - th, j_1 - th$ and $i_2 - th, j_2 - th$ components of the original system respectively. As the entries vary over all possible combinations, the matrix determines the dynamics of $(X \times X, f \times f)$. Thus, the system $(X \times X, f \times f)$ can be embedded into certain symbolic dynamical system of $2n$ symbols.

In this case the system (X, f) is weakly mixing if and only if M^{*2} is irreducible.

For the system $(\sum, s)$, topological entropy, defined as: $h(\sum) = \lim\limits_{N\to\infty} \frac{\log|B_N(\sum)|}{N}$; where $B_N(\sum)$ is the cardinality of all possible allowed strings in $\sum$ of length N.

For the full r-shift $\mathcal{A}^{\mathbb{Z}}$, $|B_N(\mathcal{A}^{\mathbb{Z}})| = r^N$, and so $h(\mathcal{A}^{\mathbb{Z}}) = \log r$. If $\sum_M$ is the subshift of finite type given by the transition matrix M then $h(\sum_M) = \log e_M$, where e_M is the largest eigenvalue of M. This result relies on the Perron-Frobenius Theory.

Consider the SFT $\sum$ given by the adjacency matrix: $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. A simple computation shows that e_A is the golden mean, and so the entropy of the SFT $\sum$ is the log of the golden mean.

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