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Common Fixed Point without Weak Reciprocal Continuity

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Abstract

A generalization of recent common fixed point theorem of Pant et al is proved under a generalized inequality involving two pairs of self-maps satisfying the common property E.A. with restricted completeness of the metric space.

Keywords: Weak reciprocal continuity, Common Property E.A., Weakly compatible self-maps, Coincidence point and Common fixed point

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1. INTRODUCTION

Let X be a metric space with metric d, $x \in X$ and f, a self-map on X. We write fx for the f-image of x, $f(X)$ for the range of f, and Sf for the composition $S \circ f$ of self-maps S and f on X. Self-maps f and S on X are *commuting* if $fgx = gfx$ for all $x \in X$.

It is obvious that a fixed point of any self-map f on X can always be regarded as a common fixed point of it and the identity self-map on X. Highlighting this interdependence between the commutativity and the existence common fixed point in a more general context, Jungck [8] obtained a necessary and sufficient condition for a continuous self-map on a complete metric space to have a fixed point, which is an extension of the well-known Banach contraction principle. For some more results on commuting mappings by dropping the continuity, relaxing the completeness of X and/or weakening the Jungck's contraction condition, one can refer to the works of Fisher ([3,4]), Fisher and Khan [5], Chang [6], Das and Naik [7], Pant [11] and Singh and Singh [20].

Sessa [18] introduced a weaker condition of commutativity as given below:

***Definition* 1.1** Self-maps S and f on X are *weakly commuting* if

$$d(Sfx, fSx) \le d(Sx, fx) \quad \text{for all } x \in X. \quad \ldots \quad (1)$$

Obviously, every commuting pair is weakly commuting. The converse is not true [18]. This was further generalized by Jungck [9] in 1986 as follows:

***Definition* 1.2** Self-maps S and f on X are *compatible* or *asymptotically commuting* if

$$\lim_{n\to\infty} d\,(Sfx_n, fSx_n) = 0 \qquad \ldots \qquad (2)$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t \text{ for some } t \in X. \qquad \ldots \qquad (3)$$

Note that every weakly commuting pair is compatible but the converse is not true [9]. Splitting the condition (2) in various ways, Pathak and Khan [16] in a comparative study, characterized various types of compatibility in terms of continuity. They showed that all types of compatibility for a pair of continuous self-maps are equivalent to their compatibility. While as a generalization of weak commutativity, Pant [12] introduced

Definition 1.3 Self-maps f and S on X are *R-weakly commuting* if

$$d\,(fSx, Sfx) \le R\, d(Sx, fx) \text{ for all } x\in X \text{ for some } R > 0 \qquad \ldots \qquad (4)$$

For $R = 1$, we see that (4) reduces to (2). That is, weak commutativity is a particular case of R-weak commutativity when $R = 1$. Weakly commuting maps are not R-weakly commuting in general. However, R-weak commutativity implies weak commutativity only when $R \le 1$, as shown in [12].

Splitting the condition (4) in two ways, Pathak et al [15] gave

Definition 1.4 Self-maps f and S on X are *R-weakly commuting* of type (A_g) if

$$d\,(fSx, SSx) \le R\, d(Sx, fx) \text{ for all } x\in X \text{ for some } R > 0 \qquad \ldots \qquad (5)$$

Definition 1.5 Self-maps f and S on X are *R-weakly commuting* of type (A_f) if

$$d\,(ffx, Sfx) \le R\, d(Sx, fx) \text{ for all } x\in X \text{ for some } R > 0 \qquad \ldots \qquad (6)$$

We see that Definition 1.5 is obtained from Definition 1.4 by interchanging the roles of S and f. In a comparative study of various weaker forms of commuting maps, Singh and Tomar [19] remarked that the notion of Definition 1.3 is independent of Definition 1.4.

Definition 1.6 A point $x\in X$ is a *coincidence point* of self-maps S and f if $Sx = fx = p$ and p is a *point of coincidence* of S and f w. r. t. x.

Remark 1.1 (Singh and Tomar, [19]) Compatibility, R-weak commutativity and their types implies their commutativity at their coincidence points. Self-maps S and f which commute at their coincidence points are called *weakly compatible* [10], partially commuting [17] or compatible type (N) [21].

The following is an easy consequence for weakly compatible maps:

Remark 1.2 If self-maps S and f are weakly compatible, then their point of coincidence w. r. t. a coincidence point will also be a coincidence point for them.

It can be easily seen that compatibility and noncompatibility are included in the class of all pairs of self-maps with the choice (3) for some sequence $\langle x_n \rangle_{n=1}^{\infty}$.

In fact, we have

Definition 1.7 (Aamri and Moutawakil, [1]). Two self-maps S and f on X satisfy the *property E. A.* if there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X with the choice (3).

Remark 1.3 (Pathak et al [15]): Weak compatibility and the property E. A. are independent of each other.

The following example gives self-maps S and f having the property E. A. but not satisfying the inclusion $S(X) \subset f(X)$.

Example 1.1 Let $X = [-1,1]$ with the usual metric d.

Define $Sx = \begin{cases} \frac{1}{2} & \text{if } \quad x = -1 \\ \frac{x}{4} & \text{if } \ -1 < x < 1 \\ \frac{3}{5} & \text{if } \quad x = 1, \end{cases}$ and $fx = \begin{cases} \frac{1}{2} & \text{if } \quad x = -1 \\ \frac{x}{2} & \text{if } \ -1 < x < 1 \\ -\frac{1}{2} & \text{if } \quad x = 1. \end{cases}$

Set $x_n = \frac{1}{n}$ for all n = 1, 2, 3, ….. Then $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = 0$, that is S and f satisfy the property E.A. But $S(X) = \left\{\frac{1}{2}, \frac{3}{5}\right\} \cup \left(-\frac{1}{4}, \frac{1}{4}\right) \not\subset \left[-\frac{1}{2}, \frac{1}{2}\right] = f(X)$. Thus the containment pattern of range of map into the range of the other is required in common fixed point considerations.

In the study of fixed points for discontinuous maps, the following notions were introduced:

***Definition* 1.8** (Pant, [13]). Self-maps S and f on X are *reciprocally continuous* at a point $t \in X$ if for any $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with the choice (3), we have $\lim_{n\to\infty} fSx_n = ft$ and $\lim_{n\to\infty} Sfx_n = St$, and S and f are *reciprocally continuous* if and only if they are reciprocally continuous at each $t \in X$.

***Definition* 1.9** (Pant et al [14]). Self-maps S and f on X are *weakly reciprocally continuous* at $t \in X$ if $\lim_{n\to\infty} fSx_n = ft$ or $\lim_{n\to\infty} Sfx_n = St$ for any $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with choice (3), and S and f are *weakly reciprocally continuous* if they are weakly reciprocally continuous at each point of X.

Any pair of continuous maps will be a reciprocally continuous, and reciprocally continuous maps are obviously weakly reciprocally continuous but neither of the reverse implications is true ([13, 14]).

It may be possible that self-maps S and f the maps are vacuously reciprocally continuous even if they do not satisfy property E.A.

Example 1.2 Consider $S, f : \mathbb{R} \to \mathbb{R}$ with the usual metric $d(x,y) = |x - y|$, given by $Sx = ax$ and $fx = ax + b$, $x \in \mathbb{R}$, where $0 < a < 2$, and $b \neq 0$. Then $d(Sx, fx) = |b| > 0$ and $d(Sfx, fSx) = |b||a-1|$. Thus $d(Sfx, fSx) < d(Sx, fx)$ for all $x \in \mathbb{R}$. Note that S and f do not satisfy property E. A., since $d(fx, Sx) = |b| \neq 0$ for all x and for no sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$, condition (3) holds good.

The following example ensures that self-maps S and f can satisfy property E. A. without being *nonvacuously* weakly reciprocally continuous.

Example 1.3 Let $X = [2, 20]$ with the usual metric d. Define S, f: $X \to X$ be given by $S2 = 2, Sx = 6$ for $2 < x \leq 5$, $Sx = \dfrac{x+5}{5}$ for $x > 5$ and set $x_n = 5 + \dfrac{1}{n}$ for n = 1, 2, 3, …

Then $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = 2$ so that S and f satisfy property E.A. But

$$\lim_{n\to\infty} Sfx_n = \lim_{n\to\infty} S\left(2 + \frac{1}{5n}\right) = 6 \neq S2$$

and

$$\lim_{n\to\infty} fSx_n = \lim_{n\to\infty} f\left(2 + \frac{1}{5n}\right) = 12 \neq f2 .$$

Thus S and f are not (nonvacuously) weakly reciprocally continuous. In other words, given a pair of self-maps, property E. A. is weaker than nonvacuous weak reciprocal continuity.

With these ideas, the following theorem was proved in [14]:

Theorem 1.1: *Let f and S be weakly reciprocally continuous self-maps on a complete metric space X satisfying the inclusion* $S(X) \subset f(X)$ *and the inequality*

$$d(Sx, Sy) \le ad(fx, fy) + bd(fx, Sx) + c\, d(fy, Sy) \text{ for all } x, y \in X \qquad \ldots \qquad (7)$$

where a, b, c ≥ 0 with a + b + c < 1. Suppose that S and f are either compatible or R-weakly commuting of type (A_g) *or* (A_f)*. Then f and S will possess a unique common fixed point.*

We prove a generalization of Theorem 1.1 under a generalized inequality involving two pairs of self-maps satisfying the common property E. A. under restricted completeness of the metric space.

2. Main Results and Discussion

Property E. A. was extended by Aliouche [2] to two pairs of self-maps as follows:

***Definition* 2.1** Two pairs (S, f) and (T, g) of self-maps on *X* satisfy the common property E.A. if there exist sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ in *X* such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} fx_n = \lim_{n \to \infty} Ty_n = \lim_{n \to \infty} gy_n = z \text{ for some } z \in X. \qquad \ldots \qquad (8)$$

We first prove our first result which gives a relation between the common property E. A. and property E. A. of individual pairs of self-maps:

Theorem 2.1 *Let f, g, S, and T be self-maps on a metric space X such that*

$$[1 + \beta d(fx, gy)]\, d(Sx, Ty) \le \alpha \max\left\{d(fx, gy), d(fx, Sx), d(gy, Ty), \tfrac{1}{2}[d(fx, Ty) + d(Sx, gy)]\right\}$$

$$+ \beta[d(fx, Sx)\, d(gy, Ty) + d(fx, Ty)\, d(Sx, gy)] \text{ for all } x, y \in X \quad \ldots \qquad (9)$$ *where* $0 \le \alpha < 1$, and $\beta \ge 0$.

Suppose one of the following statements holds good:

(a) *the pair* (S, f) *satisfies the property E. A. and* $S(X) \subset g(X)$

(b) *the pair* (T, g) *satisfies the property E. A. and* $T(X) \subset f(X)$.

Then the pairs (S, f) *and* (T, g) *satisfy the common property E.A.*

Proof. Suppose that (a) holds good. That is (S, f) satisfies the property E. A.

Then there exist points $x_1, x_2, \ldots$ in *X* such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} fx_n = z \text{ for some } z \in X. \qquad \ldots \qquad (10)$$

In view of the inclusion, $S(X) \subset g(X)$, choose $\langle y_n \rangle_{n=1}^{\infty} \subset X$ such that $Sx_n = gy_n$. Then from (10), it follows that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = z$. We prove that $\lim_{n\to\infty} Ty_n = z$. In fact, writing $x = x_n$ and $y = y_n$ in (9), we get

$$[1+\beta d(fx_n, gy_n)]\, d(Sx_n, Ty_n) \le \alpha \max\{d(fx_n, gy_n), d(fx_n, Sx_n), d(gy_n, Ty_n), \tfrac{1}{2}[d(fx_n, Ty_n) + d(Sx_n, gy_n)]\} + \beta[d(fx_n, Sx_n)\, d(gy_n, Ty_n) + d(fx_n, Ty_n)\, d(Sx_n, gy_n)]$$

Applying the limit as $n \to \infty$ and then using (10) and the continuity of the metric d, we see that $[1+\beta.0]\, d(z, \lim_{n\to\infty} Ty_n) \le \alpha \max\{0, 0, d(z, \lim_{n\to\infty} Ty_n), \tfrac{1}{2} d(z, \lim_{n\to\infty} Ty_n)\} + \beta[0+0]$

so that $d(z, \lim_{n\to\infty} Ty_n) \le \alpha d(z, \lim_{n\to\infty} Ty_n)\cdot$ Thus $d(z, \lim_{n\to\infty} Ty_n) = 0$. Thus

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gy_n = z, \quad \ldots \quad (11)$$

showing that (S, f) and (T, g) satisfy the common property E.A.

On the other hand suppose that (b) holds good. Then there exist points $y_1, y_2, \ldots$ in X such that $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gy_n = z$ for some $z \in X$. Since, $T(X) \subset f(X)$, we get $Ty_n = fx_n$ for some sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X so that $\lim_{n\to\infty} fx_n = z$. Taking $x = x_n$ and $y = y_n$ in (9), applying the limit as $n \to \infty$, and proceeding as in the earlier steps, we get (11). Thus (S, f) and (T, g) satisfy the common property E. A. ■

Our next result is

Theorem 2.2 *Let f, g, S, and T be two pairs of self-maps on a metric space X satisfying the inequality* (9), *and the common property E. A. Then*

(c) (S, f) *has a coincidence point, provided* $f(X)$ *is complete.*

In addition, if $S(X) \subset g(X)$, *then* (T, g) *will also have a coincidence point.*

(d) (T, g) *has a coincidence point, provided* $g(X)$ *is complete.*

In addition, if $T(X) \subset f(X)$, *then* (S, f) *will also have a coincidence point.*

Further if

(e) (S, f) *and* (T, g) *are weakly compatible,*

then all the four maps f, g, S, and T have a common coincidence point which will in turn be their unique common fixed point.

Proof. Let *f*, *g*, *S*, and *T* satisfy the common property E.A. Then there exist sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ in *X* with the choice (8) or (11).

First suppose that $f(X)$ is complete subspace of *X*. Then $fr = z$ for some $r \in X$.

Now taking $x = r$ and $y = y_n$ in (9), we get

$$[1+\beta d(fr, gy_n)]d(Sr, Ty_n) \le \alpha \max\left\{d(fr, gy_n), d(fr, Sr), d(gy_n, Ty_n), \tfrac{d(fr,Ty_n)+d(Sr,gy_n)}{2}\right\}$$
$$+\beta[d(fr, Sr)\,d(gy_n, Ty_n) + d(fr, Ty)\,d(Sr, gy_n)].$$

In the limiting case as $n \to \infty$, this with $fr = z$ and (11) gives

$[1+\beta.0]d(Sr, fr) \le \alpha \max\left\{0, d(fr, Sr), 0, \tfrac{1}{2}d(Sr, fr)\right\} + \beta.0$ or $d(Sr, fr) \le \alpha d(Sr, fr)$ so that $d(Sr, fr) = 0$ or $Sr = fr$.

Further since $S(X) \subset g(X)$, we can find some $v \in X$ such that $Sr = gv$. Then writing $x = r$ and $y = v$ in (9), we get

$$[1+\beta d(fr, gv)]d(Sr, Tv) \le \alpha \max\left\{d(fr, gv), d(fr, Sr), d(gv, Tv), \tfrac{d(fr,Tv)+d(Sr,gv)}{2}\right\}$$
$$+\beta[d(fr, Sr)\,d(gv, Tv) + d(fr, Tv)\,d(Sr, gv)].$$

Using $Sr = fr$ and $Sr = gv$, this gives $[1+\beta.0]d(gv, Tv) \le \alpha \max\left\{d(gv, Tv), \tfrac{d(gv,Tv)}{2}\right\}$ $+\beta[0+0]$ or $d(gv, Tv) \le \alpha d(gv, Tv)$ so that $gv = Tv$. That is *v* is a coincidence point of (T, g). Thus $Sr = fr = gv = Tv = w$, say. But then from Remark 1.2 it follows from that *w* is a common coincidence point for the four self-maps, that is

$$gw = Tw = fw = Sw. \qquad \ldots \qquad (12)$$

Finally writing $x = r$ and $y = w$ in (11) and then using (12), we get

$$[1+\beta d(fr, gw)]d(Sr, Tw) \le \alpha \max\left\{d(fr, gw), d(fr, Sr), d(gw, Tw), \tfrac{d(fr,Tw)+d(Sr,gw)}{2}\right\}$$
$$+\beta[d(fr, Sr)\,d(gw, Tw) + d(fr, Tw)\,d(Sr, gw)]$$

or $[1+\beta d(w, gw)]d(w, gw) \le \alpha \max\left\{d(w, gw), \tfrac{d(w,gw)+d(w,gw)}{2}\right\} + \beta d(w, gw)\,d(w, gw)$

or $d(w, gw) \le \alpha\, d(w, gw)$ so that $d(w, gw) = 0$ or $gw = w$. Thus *w* is a common fixed point of *f*, *g*, *S*, and *T*.

On the other hand, suppose that $g(X)$ is complete subspace of *X*. Then $gw = z$ for some $w \in X$. Now taking $x = x_n$ and $x = w$ in (9), we get

$$[1+\beta d(fx_n, gw)]\, d(Sx_n, Tw) \le \alpha \max\left\{d(fx_n, gw), d(fx_n, Sx_n), d(gw, Tw), \tfrac{d(fx_n, Tw)+d(Sx_n, gw)}{2}\right\}$$
$$+\beta[d(fx_n, Sx_n)\, d(gw, Tw) + d(fx_n, Tw)\, d(Sx_n, gw)].$$

As $n \to \infty$, this with $gw = z$ and (11) gives

$[1+\beta.0]\, d(gw, Tw) \le \alpha \max\left\{d(gw, Tw), \tfrac{d(gw, Tw)}{2}\right\} + \beta.0$ or $d(gw, Tw) \le \alpha d(gw, Tw)$ so that $d(gw, Tw) = 0$ or $gw = Tw$.

Now $T(X) \subset f(X)$ implies that $Tw = fp$ some $p \in X$. Writing $x = p$ and $y = w$ in (9),

$$[1+\beta d(fp, gw)]\, d(Sp, Tw) \le \alpha \max\left\{d(fp, gw), d(fp, Sp), d(gw, Tw), \tfrac{d(fp, Tw)+d(Sp, gw)}{2}\right\}$$
$$+\beta[d(fp, Sp)\, d(gw, Tw) + d(fp, Tw)\, d(Sp, gw)].$$

This and $gw = Tw = fp$ gives $[1+\beta.0]\, d(Sp, fp) \le \alpha \max\left\{d(fp, Sp), \tfrac{d(fp, Sp)}{2}\right\} + \beta.0$ or $d(fp, Sp) \le \alpha d(fp, Sp)$ so that $fp = Sp$. That is p is a coincidence point of (S, f).

Thus $gw = Tw = fp = Sp = \gamma$. This together with Remark 1.2 implies that γ is a common coincidence point for the four self-maps, which in turn can be shown to be their common fixed point as in the earlier case. The uniqueness of the common fixed point follows directly from (9) and the choice of the constant α. ■

Writing $g = f$ in Theorem 2.2, it can be easily seen that the condition (e) can be replaced with a weaker condition that either (S, f) or (T, f) is weakly compatible pair to obtain a common fixed point. Thus we have

Corollary 2.1: *Let f, S, and T be self-maps on metric space X satisfying the inclusions* $S(X) \subset f(X)$ *and* $T(X) \subset f(X)$, *and the inequality*

$$[1+\beta d(fx, fy)]\, d(Sx, Ty) \le \alpha \max\left\{d(fx, fy), d(fx, Sx), d(fy, Ty), \tfrac{d(fx, Ty)+d(Sx, fy)}{2}\right\}$$
$$+\beta[d(fx, Sx)\, d(fy, Ty) + d(fx, Ty) d(Sx, fy)]$$
$$\text{for all } x, y \in X \qquad \ldots \qquad (13)$$

where $0 \le \alpha < 1$, *and* $\beta \ge 0$. *Suppose that the common property E. A. holds good. Then* (S, f) *and* (T, f) *have a common coincidence point, provided* $f(X)$ *is complete.*

Further if

(*f*) (S, f) *or* (T, f) *is weakly compatible,* *then f, S and T have a common coincidence point which will be their unique common fixed point.*

Again in view of Theorem 2.1 with $g = f$, we can replace the common property E. A. in Corollary 2.1 with the property E. A. of either of the pairs (S, f) and (T, f), we have

Corollary 2.2: *Let f, S, and T be self-maps on metric space X such that* $S(X) \subset f(X)$ *and the inequality* (13). *Suppose that one of the pairs* (S, f) and (T, f) *satisfies the property E. A. and the other weakly compatible. If* $f(X)$ *is complete, then f, S and T have a unique common fixed point.*

Now taking $g = f$, $T = S$ and $\beta = 0$ in Corollary 2.2, we have

Corollary 2.3: *Let f and S be self-maps on metric space X satisfying the inclusion* $S(X) \subset f(X)$ *and for all* $x, y \in X$

$$d(Sx, Ty) \le \alpha \max\left\{d(fx, fy), d(fx, Sx), d(fy, Sy), \tfrac{d(fx,Sy)+d(Sx,fy)}{2}\right\}. \qquad \ldots \qquad (14)$$

Suppose that the pair (S, f) *satisfies the property E. A. and is weakly compatible. If* $f(X)$ *is complete, then f, S and T will have a unique common fixed point.*

Assertion. Corollary 2.3 is a generalization of Theorem 1.1. In fact, note that

$$ad(fx, fy) + bd(fx, Sx) + c\,d(fy, Sy) \le \alpha \max\{d(fx, fy), d(fx, Sx), d(fy, Ty)\}$$

for all $x, y \in X$ where $\alpha = a + b + c$. But the right hand side of this inequality is still less than or equal to $\alpha \max\left\{d(fx, fy), d(fx, Sx), d(fy, Sy), \tfrac{d(fx,Sy)+d(Sx,fy)}{2}\right\}$. Thus (14) holds whenever (7) does. In other words, (14) is weaker than (7).

Let $x_0 \in X$ be arbitrary. Since $S(X) \subset f(X)$ we can choose points $x_1, x_2, \ldots$ in X inductively such that $Sx_{n-1} = fx_n$ for $n \ge 1$. From the proof given in [14], it follows that the sequence $\langle fx_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in X and hence converges to some $z \in X$. That is (S, f) satisfies the property E. A.

Hence by Corollary 2.2, f and S have a coincidence point, say p, provided $f(X)$ is complete. In view of Remark 1.2, the point of coincidence w. r. t. p becomes the unique common fixed point. Thus the conclusion of Theorem 2.2 follows from Corollary 2.2.

Here we note that neither weak reciprocal continuity nor compatibility of the maps is used. However we relaxed the completeness of the space X and employed the complete-ness of its subspace.

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