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A new approach to the generalized Lucas sequence

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ABSTRACT

The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. In this article, we reconsider the generalization $\left\{L_n^{(a,b)}\right\}$ studied earlier and defined by the recurrence relation $L_n^{(a,b)} = aL_{n-2} + bL_{n-1}$; for all $n \geq 2$; where a and b are fixed positive integers and L_n is a classic Lucas number. Here we relate this sequence with the sequence of Fibonacci numbers and produce an extended Binet's formula for $\left\{L_n^{(a,b)}\right\}$ and, thereby, identities such as Cassini's and Catalan's. Moreover, we present sum formulas concerning this generalization. We also study this sequence when the subscript n is negative.

Mathematics Subject Classification: 11B39, 11B37.

Key Words: Fibonacci numbers, Lucas numbers, Generalized Lucas numbers, Binet formula.

1. INTRODUCTION

The well-known Fibonacci sequence and the Lucas sequence are the two shining stars in the vast array of integer sequences. They have fascinated both amateurs and professional mathematicians for centuries, and they continue to charm us with their beauty, abundant applications, and ubiquitous habit of occurring in totally surprising and unrelated places. Both these sequences are famous for possessing wonderful and amazing properties and have been studied over several years. In the theory of numbers, Fibonacci sequence has always fertile the ground for mathematicians.

The Fibonacci sequence $\{F_n\}$, named after Leonardo Pisano Fibonacci (1170−1250), is defined recursively by the relation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, where $F_0 = 0$, $F_1 = 1$. This gives the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 … . This sequence arise naturally in many unexpected places and used in equally surprising places like computer algorithms [1, 7, 11], some areas of algebra [5, 6, 10], graph theory [3], quasi crystals [2, 14] and many areas of mathematics. The Fibonacci numbers also occur in Pascal's triangle [9]. They occur in a variety of other fields such as finance, art, architecture, music, etc. [8, 13] For extensive resources on Fibonacci and Lucas numbers, one can refer Koshy [9].

Also the sequence of Lucas numbers $\{L_n\}$ is defined recursively by the relation $L_n = L_{n-1} + L_{n-2}$, for all $n \geq 2$, with initial conditions $L_0 = 2$ and $L_1 = 1$.

Many kinds of generalizations of Fibonacci numbers and Lucas numbers have been presented in the literature; some by preserving the initial conditions but altering the recurrence relation and some by varying the initial conditions but preserving the recurrence relation. There are a lot of identities of Fibonacci and Lucas numbers described by Koshy [13]. In this paper, we reconsider the sequence $\left\{L_n^{(a,b)}\right\}$ defined by Das, Patel, Shah [4] which is defined by the recurrence relation

$$L_n^{(a,b)} = aL_{n-2} + bL_{n-1} \text{ ; for all } n \geq 2; \tag{1.1}$$

with $L_0 = 2, L_1 = 1$ and *a, b* are positive integers. First few terms of this generalized Lucas sequence $\{L_n^{(a,b)}\}$ are: $-a + 2b\,, 2a + b\,, a + 3b\,, 3a + 4b\,, \ldots$. They proved some curious results. Here we use different approach to prove some these results along with number of new interesting ones.

Now, it is easy to observe the identity $L_n = F_{n-1} + F_{n+1}$. This shows that any result related with the Lucas number can be converted in to the result containing Fibonacci number. This identity triggered the idea of relating $L_n^{(a,b)}$ with F_n which ultimately resulted in this article.

Here for convenience, throughout we write $L_n^{(a,b)} = G_n$, when $a\,, b$ are fixed positive integers. Then the sequence $\{G_n\}$ is defined by the recurrence relation

$$G_n = aL_{n-2} + bL_{n-1} \text{ ; for all } n \geq 2; \tag{1.2}$$

with $L_0 = 2, L_1 = 1$ and *a, b* are nonzero real numbers.

We first derive an identity relating G_n and F_n , which will be used extensively throughout the article.

Proposition 1.1: $G_n = (3a - b)F_{n-1} + (2b - a)F_n$.

Proof: We have $G_n = aL_{n-2} + bL_{n-1}$. Now since $L_n = F_{n-1} + F_{n+1}$, we have

$$\begin{aligned} G_n &= a(F_{n-3} + F_{n-1}) + b(F_{n-2} + F_n) \\ &= a(F_{n-1} - F_{n-2} + F_{n-1}) + b(F_{n-2} + F_n) \\ &= 2aF_{n-1} + (b - a)F_{n-2} + bF_n \end{aligned}$$

Thus, $G_n = (3a - b)F_{n-1} + (2b - a)F_n$.

We use it to prove that the terms of the sequence $\{G_n\}$ satisfies the Fibonacci recurrence relation.

Proposition 1.2: $G_n = G_{n-1} + G_{n-2}$.

Proof: We have $G_n = (3a - b)F_{n-1} + (2b - a)F_n$. But since $F_n = F_{n-1} + F_{n-2}$, we have

$$\begin{aligned} G_n &= (3a - b)(F_{n-2} + F_{n-3}) + (2b - a)(F_{n-1} + F_{n-2}) \\ &= \{(3a - b)F_{n-2} + (2b - a)F_{n-1}\} + \{(3a - b)F_{n-3} + (2b - a)F_{n-2}\} \end{aligned}$$

Thus, $G_n = G_{n-1} + G_{n-2}$.

2. SOME SUMMATION FORMULAE

We first derive some elementary summation formula for the terms of the sequence $\{G_n\}$.

Proposition 2.1: $\sum_{i=1}^{n} G_i = G_{n+2} - (2a+b)$.

Proof: We have $G_n = (3a-b)F_{n-1} + (2b-a)F_n$. This gives

$$\sum_{i=1}^{n} G_i = (3a-b)\sum_{i=1}^{n} F_{i-1} + (2b-a)\sum_{i=1}^{n} F_i .$$

Now it is known that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$. (See: Koshy [9])

Thus, $\sum_{i=1}^{n} G_i = (3a-b)(F_{n+1}-1) + (2b-a)(F_{n+2}-1)$

This on simplification gives $\sum_{i=1}^{n} G_i = G_{n+2} - (2a+b)$.

We now derive the result for the sum of first *n* terms of sequence $\{G_n\}$ with odd subscripts.

Proposition 2.2: $\sum_{i=1}^{n} G_{2i-1} = G_{2n} - G_0$.

Proof: We have $G_{2n-1} = (3a-b)F_{2n-2} + (2b-a)F_{2n-1}$. This gives

$$\sum_{i=1}^{n} G_{2i-1} = (3a-b)\sum_{i=1}^{n} F_{2(i-1)} + (2b-a)\sum_{i=1}^{n} F_{2i-1}.$$

But since by Koshy [9], $\sum_{i=1}^{n} F_{2(i-1)} = F_{2n-1} - 1$ and $\sum_{i=1}^{n} F_{2i-1} = F_{2n}$, we have

$$\sum_{i=1}^{n} G_{2i-1} = (3a-b)(F_{2n-1}-1) + (2b-a)F_{2n} = (3a-b)F_{2n-1} + (2b-a)F_{2n} - 3a + b.$$

This proves that $\sum_{i=1}^{n} G_{2i-1} = G_{2n} - G_0$.

We next obtain similar result for the even subscripts.

Proposition 2.3: $\sum_{i=1}^{n} G_{2i} = G_{2n+1} - G_1$.

Proof: We have $G_{2n} = (3a-b)F_{2n-1} + (2b-a)F_{2n}$. This gives

$$\sum_{i=1}^{n} G_{2i} = (3a-b)\sum_{i=1}^{n} F_{2i-1} + (2b-a)\sum_{i=1}^{n} F_{2i} .$$

Again from Koshy [9] we know that $\sum_{i=1}^{n} F_{2i-1} = F_{2n}$ and $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$. Thus, we get

$$\sum_{i=1}^{n} G_{2i} = (3a-b)F_{2n} + (2b-a)(F_{2n+1}-1) = (3a-b)F_{2n} + (2b-a)F_{2n+1} + a - 2b.$$

Thus, $\sum_{i=1}^{n} G_{2i} = G_{2n+1} - G_1$.

3. EXTENDED BINET'S FORMULA FOR G_n

In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers (See: Vajda [13]). It is well-known that both theses Binet formulae are used commonly in the Fibonacci numbers theory. Here we derive the extended Binet formula for G_n expressed as a function of the roots α and β of the charactersictic equation associated to the recurrence relation for the Fibonacci numbers:

$$x^2 - x - 1 = 0. \tag{3.1}$$

The Binet formula for Fibonacci and Lucas numbers are respectively given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \text{ and } L_n = \alpha^n + \beta^n = \left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\},$$

where $\alpha = \left(\frac{1+\sqrt{5}}{2}\right)$ is famously referred as '*golden ratio*'. Here we note that $\alpha > \beta$, $\alpha - \beta = \sqrt{5}$ and $\alpha\beta = -1$.

Theorem 3.1: (*Extended Binet formula*) The n[th] term of sequence $\{G_n\}$ is given by

$$G_n = \frac{e\alpha^n - f\beta^n}{\alpha-\beta}, \tag{3.2}$$

where $e = \beta(b-3a) + 2b - a$, $f = \alpha(b-3a) + 2b - a$.

Proof: We have $G_n = (3a-b)F_{n-1} + (2b-a)F_n$.

Using Binet's formula for F_n, we get

$$G_n = (3a-b)\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right) + (2b-a)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right).$$

$\therefore \sqrt{5}G_n = \alpha^n\left(\frac{3a}{\alpha} - \frac{b}{\alpha} + 2b - a\right) - \beta^n\left(\frac{3a}{\beta} - \frac{b}{\beta} + 2b - a\right).$

But since $\alpha\beta = -1$, we have

$\sqrt{5}G_n = \alpha^n(-3a\beta + b\beta + 2b - a) - \beta^n(-3a\alpha + b\alpha + 2b - a)$

$\quad = \alpha^n\{\beta(b-3a) + 2b - a\} - \beta^n\{\alpha(b\alpha - 3a) + 2b - a\}.$

If we write $e = \beta(b-3a) + 2b - a$ and $f = \alpha(b-3a) + 2b - a$, we get $G_n = \frac{e\alpha^n - f\beta^n}{\alpha-\beta}$.

Remark: $ef = \{\beta(b-3a) + 2b - a\}\{\alpha(b-3a) + 2b - a\}$

$\quad = -(b-3a)^2 + (2b-a)^2 + (2b-a)(b-3a)$

$\quad = -5(a^2 + ab - b^2).$

We call this constant as the *characteristic* of the sequence $\{G_n\}$ and write $\mu = ef$.

Proposition 3.2: (1) $e = \sqrt{5}c$ (2) $f = -\sqrt{5}d$.

Proof: (1) We have $\sqrt{5}c = (\alpha-\beta)\{a + (a-b)\beta\}$

$\quad = \alpha a - \beta a + \alpha\beta(a-b) - \beta^2(a-b)$

$\quad = \alpha a - \beta a - (1+\beta)(a-b) - (a-b)$

$\quad = b\beta - 3a\beta + b + (b-a)$

$\quad = \beta(b-3a) + 2b - a = e.$

(2) We have $-\sqrt{5}d = -(\alpha-\beta)\{a + (a-b)\alpha\}$

$\quad = a\beta - \alpha a + \alpha\beta(a-b) - \alpha^2(a-b)$

$\quad = a\beta - \beta a - (1+\alpha)(a-b) - (a-b)$

$\quad = \alpha b - 3a\alpha + 2b - a$

$\quad = \alpha(b-3a) + 2b - a = f.$

Corollary 3.3: $ef = -5cd$.

Proof: Follows immediately from above lemma.

Corollary 3.4: $G_n = L_n^{(a,b)}$.

Proof: $G_n = \frac{e\alpha^n - f\beta^n}{\alpha-\beta} = \frac{e}{\alpha-\beta}\alpha^n + \frac{-f}{\alpha-\beta}\beta^n = \frac{e}{\sqrt{5}}\alpha^n + \frac{-f}{\sqrt{5}}\beta^n.$

But by above lemma, we have $\frac{e}{\sqrt{5}} = c$ and $\frac{-f}{\sqrt{5}} = d$. Thus, $G_n = c\alpha^n + d\beta^n$.

We next use (3.2) to show that limiting ratio of any two consecutive terms of this sequence converge to a fixed real number.

Proposition 3.5: $\lim_{n\to\infty}\frac{G_{n+1}}{G_n}=\alpha$.

Proof: Using (3.2), we have $\lim_{n\to\infty}\frac{G_{n+1}}{G_n}=\lim_{n\to\infty}\frac{e\alpha^{n+1}-f\beta^{n+1}}{e\alpha^{n}-f\beta^{n}}$. Then,

$$\lim_{n\to\infty}\frac{G_{n+1}}{G_n}=\lim_{n\to\infty}\frac{e\alpha-f\left(\frac{\beta}{\alpha}\right)^n\beta}{e-f\left(\frac{\beta}{\alpha}\right)^n}.$$

But we know that $\beta=\frac{1-\sqrt{5}}{2}<\frac{1+\sqrt{5}}{2}=\alpha$. Thus, $\frac{\beta}{\alpha}<1\Rightarrow\left(\frac{\beta}{\alpha}\right)^n\to 0$, as $n\to\infty$.

Thus $\lim_{n\to\infty}\frac{G_{n+1}}{G_n}=\alpha$.

The following interesting result follows for the powers of α and β.

We now use (3.2) to prove some interesting identities for this sequence. In the following Lemma, we derive the *extended Cassini's identity* for G_n which connects three consecutive G_n's together.

Proposition 3.6: [*Extended Cassini's identity*]

$$G_{n+1}G_{n-1}-G_n^2=\mu(-1)^n.$$

Proof: Using (3.2), we have

$$\begin{aligned}5(G_{n+1}G_{n-1}-G_n^2)&=(e\alpha^{n+1}-f\beta^{n+1})(e\alpha^{n-1}-f\beta^{n-1})-(e\alpha^n-f\beta^n)^2\\&=-ef(\alpha^{n+1}\beta^{n-1}+\alpha^{n-1}\beta^{n+1})+2ef(-1)^n\\&=-\mu(\alpha\beta)^{n-1}(\alpha^2+\beta^2)+2\mu(-1)^n.\end{aligned}$$

Since $L_n=\alpha^n+\beta^n$ and $L_2=3$, we have

$$G_{n+1}G_{n-1}-G_n^2=-\mu(-1)^{n-1}(3)+2\mu(-1)^n.$$

Hence, $G_{n+1}G_{n-1}-G_n^2=5\mu(-1)^n$, as required.

We next prove more generalized form of extended Cassini's identity which connects three consecutive G_n's with suffixes in arithmetic progression for fixed *n*.

Proposition 3.7: [*Extended Catalan's identity*]

$$G_n^2-G_{n+r}G_{n-r}=\frac{1}{5}ef(-1)^{n+1}(2-L_{2r}).$$

Proof: Using (3.2), we have

$$\begin{aligned}5(G_n^2-G_{n+r}G_{n-r})&=(e\alpha^n-f\beta^n)^2-(e\alpha^{n+r}-f\beta^{n+r})(e\alpha^{n-r}-f\beta^{n-r})\\&=-2ef(\alpha\beta)^n+ef(\alpha\beta)^n\alpha^r(-\alpha)^r+ef(\alpha\beta)^n\beta^r(-\beta)^r\\&=-2ef(-1)^n+ef(-1)^n\alpha^r(-\alpha)^r+ef(-1)^n\beta^r(-\beta)^r\\&=-ef(-1)^n(2+\alpha^{2r}+\beta^{2r})\\&=ef(-1)^{n+1}(2+\alpha^{2r}+\beta^{2r}).\end{aligned}$$

Hence, $G_n^2-G_{n+r}G_{n-r}=\frac{1}{5}ef(-1)^{n+1}(2-L_{2r})$.

The next result shows how (3.2) is useful to express G_{n+1} in terms of G_n.

Proposition 3.8: $G_{n+1} = \alpha G_n + f\beta^n$, where $e = \beta(b-3a) + 2b - a$ and $f = \alpha(b-3a) + 2b - a$.

Proof: We have $\sqrt{5}G_n = e\alpha^n - f\beta^n$. Then,

$$\sqrt{5}\alpha G_n = e\alpha^{n+1} - f\alpha\beta^n$$

$$= e\alpha^{n+1} + f\beta^{n-1}$$

$$= (e\alpha^{n+1} - f\beta^{n+1}) + f(\beta^{n-1} + \beta^{n+1})$$

$$= (e\alpha^{n+1} - f\beta^{n+1}) + f\beta^{n-1}(\beta^2 + 1)$$

$$= (e\alpha^{n+1} - f\beta^{n+1}) + f\beta^{n-1}(-\sqrt{5}\beta).$$

$$\therefore \alpha G_n = \frac{e\alpha^{n+1} - f\beta^{n+1}}{\alpha - \beta} - f\beta^n = G_{n+1} - f\beta^n.$$

Thus, $G_{n+1} = \alpha G_n + f\beta^n$.

Following result follows immediately from this lemma by taking $n \to \infty$ and keeping in mind that $|\beta| < 1$.

Corollary 3.9: $G_{n+1} \approx \alpha G_n$.

4. G_n WITH NEGATIVE SUBSCRIPTS

We now extend the sequence $\{G_n\}$ backward with negative subscripts. In fact if we try to extend the sequence $\{G_n\}$ backwards, still keeping to the rule (1.2), we get the following:

n	G_n
⋮	⋮
-3	$-11a + 7b$
-2	$7a - 4b$
-1	$-4a + 3b$
0	$3a - b$
1	$-a + 2b$
2	$2a + b$
3	$a + 3b$
⋮	⋮

We can now consider G_n being defined for all integer values of *n*, both positive and negative and the sequence $\{G_n\}$ extending infinitely in both the positive and negative directions. We observe here that $G_{-n} = (-1)^n\{(3a-b)F_{n+1} - (2b-a)F_n\}$. We prove this result in the following theorem.

Theorem 4.1: $G_{-n} = (-1)^n\{(3a-b)F_{n+1} - (2b-a)F_n\}$.

Proof: We have $G_n = (3a-b)F_{n-1} + (2b-a)F_n$. Now considering $(-n)$ in place of n, we get

$$G_{-n} = (3a-b)F_{-n-1} + (2b-a)F_{-n} = (3a-b)F_{-(n+1)} + (2b-a)F_{-n} .$$

But by Koshy [9] we know that $F_{-n} = (-1)^{n+1}F_n$. Using this we have

$$G_{-n} = (3a-b)(-1)^{n+2}F_{n+1} + (2b-a)(-1)^{n+1}F_n .$$

Thus, $G_{-n} = (-1)^n\{(3a-b)F_{n+1} - (2b-a)F_n\}$.

Next we derive two important identities.

Proposition 4.2: $G_{m+n} = (3a-b)F_{m+n-1} + (2b-a)F_{m+n}$.

Proof: We prove the result by the principle of mathematical induction.

For $n = 1$, we have $G_{m+1} = (3a-b)F_m + (2b-a)F_{m+1}$, which is obviously true. Now assume that the result holds for all positive integers up to some positive integer k and we will show that it also holds for $n = k+1$ also. Now by theorem 1.2, we have

$$G_{m+k+1} = G_{m+k} + G_{m+k-1} .$$

Using theorem 1.1, we have

$$G_{m+k+1} = (3a-b)F_{m+k-1} + (2b-a)F_{m+k} + (3a-b)F_{m+k-2} + (2b-a)F_{m+k-1}$$

$$= (3a-b)(F_{m+k-1} + F_{m+k-2}) + (2b-a)(F_{m+k} + F_{m+k-1}) .$$

$$\therefore G_{m+k+1} = (3a-b)F_{m+k} + (2b-a)F_{m+k+1}.$$

Thus, by the principal of mathematical induction, result is true for every positive integer n.

Proposition 4.3: $G_{m-n} = (-1)^{n-m}\{(3a-b)F_{n-m+1} - (2b-a)F_{n-m}\}$.

Proof: From the above lemma, we have $G_{m+n} = (3a-b)F_{m+n-1} + (2b-a)F_{m+n}$.

Now considering $(-n)$ in place of n and keeping *m* as it is, we get

$$G_{m-n} = (3a-b)F_{m-n-1} + (2b-a)F_{m-n}$$

$$= (3a-b)F_{-(n-m+1)} + (2b-a)F_{-(n-m)}$$

$$= (3a-b)(-1)^{n-m+2}F_{n-m+1} + (2b-a)F_{n-m} .$$

Thus, $G_{m-n} = (-1)^{n-m}\{(3a-b)F_{n-m+1} - (2b-a)F_{n-m}\}$.

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