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An algorithm to construct Sublime Numbers

Pallavi Pathak

Jawahar Pathak

Department of Mathematical Sciences

Lincoln University, PA 19352

jpathak@lincoln.edu

Abstract

In this paper we characterize even sublime numbers in theorem 3.2. Let $p, l_1, \ldots, l_{p-1}$ be primes such that $q = 2^p - 1$, $2^q - 1$, $m_1 = 2^{l_1} - 1, \ldots, m_{p-1} = 2^{l_{p-1}} - 1$ are all distinct primes and $l_1 + \ldots + l_{p-1} = q - 1$. Then $n = 2^{q-1} \cdot m_1 \cdots m_{p-1}$ is an even sublime number. Further, every even sublime number n is of the form $2^k m$ for some odd integer m with $\sigma(n)$ and $\tau(n)$ both even, $\sigma(m) = 2^{q-1}$ and $\tau(m) = 2^{p-1}$. We use this theorem to design a constructive algorithm to generate sublime numbers.

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1 Introduction

A natural number n is called **perfect** if the sum of the divisors of n excluding n itself is equal to n. The first two such numbers are $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$. Perfect numbers have a very interesting history (see **[2]** and **[4]**) and it goes back to the days of Pythagoras. The first algorithm to construct a perfect number appeared in Euclid's Elements (see **[4]**) which is as follows:

If as many numbers as we please beginning from a unit be set out continuously in double proportion, until the sum of all becomes a prime, and if the sum multiplied into the last make some number, the product will be perfect. [Elements-IX, Proposition 36].

For example, $1 + 2 + 4 = 7$ is a prime integer. Now the sum is 7 and the last number is 4, therefore the product is $(7)(4) = 28$, is perfect. Similarly $1 + 2 + 4 + 8 + 16 = 31$ is prime, and $31 \cdot 16 = 496$ is perfect. In modern terminology this result can be stated as:

If for some $k > 1$, $2^k - 1$ *is prime then* $2^{k-1}(2^k - 1)$ *is a perfect number.*

A **sublime** number is a natural number which has a perfect number of positive divisors (including itself), and whose positive divisors add up to another perfect number. In this paper we provide an algorithm to find even sublime numbers. In section 2, we set up notations and record some known results. Main theorem is proved in section 3 and an application of the main theorem is given in section 4.

2 Notations

Most material in this section is known and can be found in any standard text on Number Theory, for example see **[1]**, chapter 11, section 2 or **[5]** chapter 2, project 3 on page 56.

Definition 2.1. For any natural number n, $\sigma(n)$ denotes the sum of distinct divisors of n and $\tau(n)$ denotes the number of distinct divisors of n.

If n is perfect, then $\sigma(n) = 2n$. For any prime integer p, $\sigma(p) = p + 1$ and $\tau(p) = 2$. Further, $\sigma(p^k) = 1 + p^2 + \ldots + p^k = \frac{p^{k+1}-1}{p-1}$. In particular, $\sigma(2^k) = 2^{k+1} - 1$. Functions like σ and τ are called arithmatic functions. They enjoy some very interesting properrties, see **[2]**, **[5]** or **[6]**for the explicit description of these properties. We will assume these properties. One worth mentioning property is **multiplicative** nature of these functions that is, for two co-prime integers m and n,

$$\sigma(mn) = \sigma(m)\sigma(n) \text{ and } \tau(mn) = \tau(m)\tau(n)$$

Next, we record some known results. Their proofs are included mainly because they are used in our main theorem. These and sinilar results can be found in **[2]** or **[5]**.

Proposition 2.1. *For any natural number p, a necessary, but not sufficient condition for $2^p - 1$ to be a prime is that p be a prime.*

Proof. Suppose p is composite and let $p = m \cdot n$, where $m, n \in \mathbb{N}, m > 1, n > 1$. Then

$$2^p - 1 = 2^{m \cdot n} - 1 = (2^m - 1)(2^{m(n-1)} + 2^{(m)(n-1)-1} + \ldots + 1),$$

which shows that $(2^m - 1)$ divides $(2^p - 1)$ and $2^p - 1$ is not a prime. When $p = 11$, $2^{11} - 1 = 23 \cdot 89$ is not a prime, which shows that the condition is not sufficient. □

Primes of the form $2^p - 1$ are called **Mersenne** primes which are discussed in **[3]**.

Theorem 2.2. *A perfect number n is even if and only if n is of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a prime.*

Proof. Suppose that $n = 2^{p-1}(2^p - 1)$, where $(2^p - 1)$ is a prime integer. Then,

$$\sigma(n) = \sigma(2^{p-1})(2^p - 1) = \sigma(2^{p-1})\sigma(2^p - 1)$$
$$= (2^p - 1)2^p = 2n.$$

Therefore n is perfect. Conversely, suppose n is an even perfect number, then we can write $n = 2^k \cdot m$, where $k \geq 1$ and m is odd. Since n is perfect,

$$\sigma(n) = 2n = 2(2^k m) = 2^{k+1} m.$$

Using the multiplicative property of σ,

$$\sigma(n) = \sigma(2^k)\sigma(m) = (2^{k+1} - 1)\sigma(m).$$

Hence, $2^{k+1}m = (2^{k+1} - 1)\sigma(m)$ and 2^{k+1} divides $\sigma(m)$. Then $\sigma(m) = 2^{k+1} \cdot d$, for some $d \in N$, $2^{k+1}m = (2^{k+1} - 1)2^{k+1}d$ and $m = (2^{k+1} - 1)d$.
We claim that d = 1. If $d > 1$, then m has at least three distinct positive divisors, namely, 1, m and d. Therefore,

$$\sigma(m) \geq m + d + 1 = (2^{k+1} - 1)d + d + 1 = 2^{k+1}d + 1,$$

But $\sigma(m) = 2^{k+1} \cdot d$ and so can not be larger than $2^{k+1} \cdot d + 1$. Hence d must be 1, $m = 2^{k+1} - 1$ and $\sigma(m) = 2^{k+1}$. Therefore, $\sigma(m) = m + 1$ and m is prime. Let $p = k + 1$ then, $n = 2^{p-1}m = 2^{p-1}(2^p - 1)$. □

Corollary. *Suppose that n is a perfect number other than 6 with the prime factorization* $n = \prod_1^r p_i^{\alpha_i}$. *Then at most one* α_i *can be equal to one.*

Proof. Suppose $n = pqm$ where p, q are distinct primes and p, q, $m)$ are pairwise co-prime. Since n is perfect, $\sigma(n) = 2n$. Therefore both $(p + 1)$ and $(q + 1)$ divide $2n$. In other words, n is an even perfect number or $n = 2^{i-1}(2^i - 1)$, with $2^i - 1$ a prime. But then the only choice is $p = 2$, $q = 3$ and $n = 6$. □

3 Sublime numbers

Definition 3.1. A natural number n is called **Sublime Number** if $\sigma(n)$ and $\tau(n)$ are both perfect numbers.

Example 3.1. 12 is a sublime number. Factors of 12 are 1, 2, 3, 4, 6 and 12. Therefore, $\sigma(12) = 28$ and $\tau(12) = 6$. Further, 6 and 28 are both perfect numbers.

Lemma 3.1. *If $m = p_1 p_2 \cdots p_r$ where p_i are distinct primes. Then*
$\sigma(m) = (p_1 + 1)(p_2 + 1) \cdots (p_r + 1)$ *and* $\tau(m) = 2^r$.

Proof. A direct consequence of the multiplicative property of σ and τ and remark following definition 2.1 proves this lemma. □

Theorem 3.2. *Let p, $l_1, l_2, \ldots, l_{p-1}$ be primes such that $q = 2^p - 1$, $2^q - 1$, $m_1 = 2^{l_1} - 1$, $\ldots$, $m_{p-1} = 2^{l_{p-1}} - 1$ are all distinct primes and $l_1 + \ldots + l_{p-1} = q - 1$. Then $n = 2^{q-1} \cdot m_1 \cdots m_{p-1}$ is an even sublime number. Further, every even sublime number n is of the form $2^k m$ with m odd, $\sigma(n)$ and $\tau(n)$ even, $\sigma(m) = 2^{q-1}$ and $\tau(m) = 2^{p-1}$.*

Proof. Let $m = m_1 \cdot m_2 \cdots m_{p-1}$, where $m_1 = 2^{l_1} - 1, \ldots$, $m_{p-1} = 2^{l_{p-1}} - 1$ are all distinct Mersenne primes. Then by Lemma 3.1, $\tau(m) = 2^{p-1}$ and

$$\begin{aligned}\sigma(m) &= (m_1 + 1) \ldots (m_{p-1} + 1) = 2^{l_1} \ldots 2^{l_{p-1}} \\ &= 2^{l_1 + \cdots + l_{p-1}} = 2^{q-1}.\end{aligned}$$

Since $n = 2^{q-1}.m$ and m is odd, we can use the multiplicative property to get,

$$\tau(n) = \tau(2^{q-1})\tau(m) = q.\tau(m) = (2^p - 1)2^{p-1}$$

and

$$\sigma(n) = \sigma(2^{q-1}.m) = \sigma(2^{q-1})\sigma(m) = (2^q - 1)2^{q-1}.$$

Hence n is an even sublime number.

Let for some odd integer m and $k \geq 1$, $n = 2^k \cdot m$ be an even sublime number with $\sigma(n)$ and $\tau(n)$ even. Then

$\sigma(n) = \sigma(2^k)\sigma(m) = (2^{k+1} - 1)\sigma(m)$

As $\sigma(n)$ is an even perfect number by theorem 2.2, $\sigma(n) = 2^{q-1}(2^q - 1)$, for some prime q.

Hence, $(2^{k+1} - 1)\sigma(m) = 2^{q-1}(2^q - 1)$. Now $2^{k+1} - 1$ is odd and $2^q - 1$ is a Mersenne Prime, therefore, $2^{k+1} - 1 = 2^q - 1$ which shows that

$$k + 1 = q \text{ and } \sigma(m) = 2^{q-1} \tag{1}$$

Therefore,

$$\tau(n) = \tau(2^k)\tau(m) = (k + 1)\tau(m) = q \cdot \tau(m) \text{ by (1)}$$

As $\tau(n)$ is perfect, $q\tau(m) = 2^{p-1}(2^p - 1)$, for some p.

Therefore $q = 2^p - 1$ and $\tau(m) = 2^{p-1}$. □

4 Application

The above result gives us an algorithm to construct a sublime number. We first rewrite the result. Suppose p is a prime with the following properties,

- $q = 2^p - 1$ is a prime.
- $2^q - 1$ is a prime.
- $q - 1$ can be partitioned into distinct primes $l_1, \ldots, l_{p-1}$ such that $m_i = 2^{l_i} - 1$ are also prime for all i.

Then $n = 2^{q-1}(\prod m_i)$ is a sublime number.

Examples

(1) Suppose $p = 2$. Then $q = 2^2 - 1 = 3$ is a prime. Further $2^q - 1 = 7$ is also a prime. Now $q - 1 = 2$, therefore $l_1 = 2$. This gives us $m_1 = 2^{l_1} - 1 = 3$. Thus we get the first sublime number $n = 2^{3-1}(3) = 12$.

(2) Suppose $p = 5$. Then $q = 2^5 - 1 = 31$. We need to find four primes l_1, l_2, l_3 and l_4. such that $\sum_{i=1}^{4} l_i = 30$ and $m_i = 2^{l_i} - 1$ are all primes. This is not possible.

(3) Suppose $p = 7$. Then $q = 2^7 - 1 = 127$ and $q - 1 = 126$.

$\sum_{i=1}^{6} l_i = 126 = 3 + 5 + 7 + 19 + 31 + 61$. Note that,

$m_1 = 2^3 - 1 = 7$,

$m_2 = 2^5 - 1 = 31$,

$m_3 = 2^7 - 1 = 127$,

$m_4 = 2^{19} - 1 = 524287$,

$m_5 = 2^{31} - 1 = 2147483647$,

$m_6 = 2^{61} - 1 = 2305843009213693951$,

are all primes. This shows that $m_i's$ are all Mersenne primes.

$\prod_{i=1}^{6} m_i = 7154711806330576349709529954736928060 1.$

and

$n = 2^{126} \prod_{i=1}^{6} m_i$

$= 6086555670238378989670371734243169622657830773351885970528324860512791691264$

To see that n is a sublime number we must show that $\sigma(n)$ and $\tau(n)$ are perfect. Now

$$\begin{aligned}\sigma(n) &= \sigma(2^{126})\sigma(7)\sigma(31)\sigma(127)\sigma(524287)\sigma(214783647)\sigma(2305843009213693951)\\ &= (2^{127}-1)(2^3)(2^5)(2^7)(2^{19})(2^{31})(2^{61})\\ &= 2^{126}(2^{127}-1)\end{aligned}$$

and

$$\begin{aligned}\tau(n) &= \tau(2^{126}\tau(7)\tau(31)\tau(127)\tau(524287)\tau(214783647)\tau(2305843009213693951)\\ &= (127)(2^6)\\ &= 2^6(2^7-1).\end{aligned}$$

which are perfect numbers.

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