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Interaction of Electromagnetism and Gravity for pp-waves Spacetimes

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Abstract

The analysis of electromagnetism in curved geometry enables one in understanding interaction of electromagnetism and gravity. Electromagnetic field equations are separable in classical theory. In general relativity (that is in a curved spacetime) they are notoriously coupled and finding a solution requires strategic approach. However, a potential formalism may have little less complexity, one of the approaches is Debye's potential formalism. Working with this directly is still difficult. Newman-Penrose formalism is a powerful geometric tool. This formalism is found to be useful in understanding electromagnetism in curved spacetimes. Earlier it was used by Hasmani and Patel (Hasmani, A.H and Patel, B.N., Electromagnetic Fields in Vaidya Spacetimes, Clifford Analysis, Clifford Algebras and Their Applications, 5(2), (2016), 171-176.) in case of Vaidya metric. In this paper we have used NP formalism to understand electromagnetic field for pp-waves spacetimes generated through Debye's potential. The spacetime under consideration is known to have non-vanishing electric and magnetic parts of the Weyl tensor and also here we have shown that non-vanishing electric and magnetic field are produced. This is useful in understanding interaction of electromagnetism and gravity.

Keywords: Debye's Potential, Newman-Penrose Formalism, pp-waves Spacetimes.

AMS Classification: 83C50, 83C60, 83C99.

1 Introduction

Maxwell's theory and general theory of relativity are well defined theories. Maxwell's theory successfully describes electromagnetism and GR describes gravity. Many attempts are made to unify these forces but in vein. Though there are some analogies between these two theories. Those analogies could be divided into two groups: (i)the kinematical analogy and (ii) dynamical (or mathematical) analogy (see [4]). Analogy between Electric and Magnetic fields in classical electrodynamics; and electric and magnetic parts of the Weyl tensor is merely mathematical, they do not have physical connection. Their relationship in particular situations of general relativity are explored. We could establish some relation between these two concepts in the case of pp-waves spacetimes.

A theoretical framework that allows to compare electromagnetism with gravitation in a fully covariant way was developed by Goulart and Falciano [4]. Harris [5] has derived equations similar to those of electromagnetic theory with assumption on speed of particle and gravitational field. In the realm of Cartan's restricted geometry, Novello [11] argued in favor of unification of electrodynamics and gravity. In order to obtain solutions of Maxwell equations in curved spacetime, Cohen and Kegeles [2] proposed approach based on application of Hertz potentials and Debye's approach, in which they obtained decoupled second order equation for a single complex potential ψ, this is an example of successful application of NP formalism [10] for this purpose. For details of NP formalism refer [1,10]. Further, Cohen et al. [3] used this formalism to study electromagnetism in Gödel spacetime. The case of Vaidya metric was studied in [8]. Continuing in the similar line of studying electromagnetism in curved spacetimes, in this paper, we have derived the equation satisfied by Debye's potential ψ for the pp-waves spacetimes which is of Petrov type N in nature. We found that there is some interaction between electromagnetism and gravity for the case of pp-waves spacetimes.

2 Newman-Penrose Formalism for pp-waves Spacetimes

Line element for pp-waves metric is

$$ds^2 = 2dudv - 2H(u,\zeta,\bar{\zeta})du^2 - 2d\zeta d\bar{\zeta}, \tag{1}$$

where u and v are real coordinates, ζ and $\bar{\zeta}$ are complex; and H is a real function of u, ζ and $\bar{\zeta}$.

For this metric, directional derivatives are

$$\begin{aligned} D =& \frac{\partial}{\partial v}, \\ \Delta =& \frac{\partial}{\partial u} + H\frac{\partial}{\partial v}, \\ \delta =& \frac{\partial}{\partial \zeta}, \\ \bar{\delta} =& \frac{\partial}{\partial \bar{\zeta}}. \end{aligned} \tag{2}$$

Using this choice of tetrad for pp-waves metric, the only non-vanishing spin coefficient is [1]

$$\nu = -H_{\zeta}. \tag{3}$$

Further, only non-vanishing complex scalar representing Weyl tensor is

$$\Psi_4 = H_{\zeta\zeta}. \tag{4}$$

For pp-waves spacetimes, non-zero components of electric and magnetic parts of the Weyl tensor are

$$E_{22} = \frac{1}{2}\frac{\partial^2 v}{\partial \zeta^2}, \tag{5}$$

$$E_{33} = \frac{1}{2}\frac{\partial^2 v}{\partial \bar{\zeta}^2}, \tag{6}$$

$$H_{22} = -\frac{1}{2}i\frac{\partial^2 v}{\partial \zeta^2}, \tag{7}$$

$$H_{33} = \frac{1}{2}i\frac{\partial^2 v}{\partial \bar{\zeta}^2}. \tag{8}$$

3 Electromagnetic Field Tensor and Debye's Potential for pp-waves Spacetimes

Electric and magnetic fields in vacuum can be expressed in terms of the vector potential $\vec{A}$ and a scalar potential U as [12]

$$\begin{aligned} \vec{E} =& -\partial_t \vec{A} - \vec{\nabla} U, \\ \vec{B} =& \vec{\nabla} \times \vec{A}, \end{aligned} \tag{9}$$

where $\vec{E}$ represents electric field and $\vec{B}$ represents magnetic field. In relativistic electrodynamics, $\vec{E}$ and $\vec{B}$ are generated be a single 4-potential $A_i = (U, \vec{A})$, this is done by an antisymmetric

[1]Some computations in this paper are done with Mathematica programs developed by Hasmani and coworkers [6, 7, 9]

the electromagnetic field tensor F_{ij} defined below

$$F_{ij} = \partial_i A_j - \partial_j A_i, \qquad \text{where } i, j = 0, 1, 2, 3 \tag{10}$$

in terms of this tensor $\vec{E}$ and $\vec{B}$ are given by

$$F^{\alpha 0} = E^{\alpha}, \quad F^{\alpha\beta} = -\epsilon^{\alpha\beta\mu} B_{\mu}, \qquad \text{where } \alpha, \beta, \mu = 1, 2, 3 \tag{11}$$

and $\epsilon_{\alpha\beta\mu}$ is Levi-Civita tensor.

Explicitly, two quantities $\vec{E}$ and $\vec{B}$ can be represented by a single tensor F_{ij} as follows

$$F_{ij} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}.$$

In classical electrodynamics, Maxwell's equations are linear. In general relativity, the physical space is a Riemannian space, also when electrodynamics is discussed in curved space it is a study of interaction of electromagnetism and gravity, hence the equations satisfied by electromagnetic fields are highly non-linear and strongly coupled. Due to strong coupling is involved, standard method for decoupling Maxwell's equations is not applicable. Another potential formalism is known as Debye's potential formalism, here a scalar (potential) ψ is used to generate electromagnetic field tensor F_{ij}. The equation satisfied by ψ in its original form is difficult to handle. In [2] Newman-Penrose formalism is used to write the decoupled equation for the Debye's complex scalar potential ψ, in terms of the intrinsic frame derivatives and the spin coefficients:

$$[(\triangle - \bar{\gamma} + \gamma + \bar{\mu})(D + 2\epsilon + \rho) - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\delta + 2\beta + \tau)]\psi = 0. \tag{12}$$

The solution of above equation is used to compute the following quantities,

$$\begin{aligned} \phi_0 =& [-(D - \epsilon + \bar{\epsilon} - \bar{\rho})(D + 2\bar{\epsilon} + \bar{\rho})]\bar{\psi}, \\ \phi_1 =& [-(D + \bar{\epsilon} + \epsilon)(\bar{\delta} + 2\bar{\beta} + \bar{\tau}) + (\pi + \bar{\tau})(D + 2\bar{\epsilon} + \bar{\rho})]\bar{\psi}, \\ \phi_2 =& [-(\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\bar{\delta} + 2\bar{\beta} + \bar{\tau}) + \lambda(D + 2\bar{\epsilon} + \bar{\rho})]\bar{\psi}. \end{aligned} \tag{13}$$

Finally, in the standard basis of the coordinate vectors, the Maxwell's Electromagnetic field tensor F_{ij} is given by

$$\begin{aligned} F_{ij} =& 2(\phi_1 + \bar{\phi_1}) n_{[i} l_{j]} + 2\phi_2 l_{[i} m_{j]} + 2\bar{\phi_2} l_{[i} \bar{m}_{j]} \\ & + 2\phi_0 \bar{m}_{[i} n_{j]} + 2\bar{\phi_0} m_{[i} n_{j]} + 2(\phi_1 - \bar{\phi_1}) m_{[i} \bar{m}_{j]}. \end{aligned} \tag{14}$$

For the metric (1), that is in the case of pp-waves spacetimes, with the help of (3) equation (12) reads as,

$$(\Delta D - \delta\bar{\delta})\psi = 0. \tag{15}$$

For the vacuumed solutions, H must have the form [13]

$$H = f(u,\zeta) + \overline{f(u,\zeta)}. \tag{16}$$

With this choice of H, the maximum allowable separability is,

$$\psi = e^{i\alpha v} F(u,\zeta,\bar{\zeta}), \tag{17}$$

where α is a separation constant and F a function which satisfies the decoupled equation obtained upon substitution of (17) in (15). Using this in equation (13), we get

$$\begin{aligned}
\phi_0 =& \alpha^2 e^{-i\alpha v} \bar{F}(u,\zeta,\bar{\zeta}),\\
\phi_1 =& i\alpha e^{-i\alpha v} \bar{F}_{\bar{\zeta}}(u,\zeta,\bar{\zeta}),\\
\phi_2 =& e^{-i\alpha v} \bar{F}_{\bar{\zeta}\bar{\zeta}}(u,\zeta,\bar{\zeta}).
\end{aligned} \tag{18}$$

These expressions give the non-zero independent components of Maxwell's electromagnetic field tensor as

$$\begin{aligned}
F_{01} =& i\alpha(e^{i\alpha v} F_{\bar{\zeta}} - e^{-i\alpha v} \bar{F}_{\bar{\zeta}}) = E_1,\\
F_{02} =& -\alpha^2 e^{-i\alpha v} H\bar{F} - e^{i\alpha v} F_{\bar{\zeta}\bar{\zeta}} = E_2,\\
F_{03} =& -\alpha^2 e^{i\alpha v} HF - e^{-i\alpha v} \bar{F}_{\bar{\zeta}\bar{\zeta}} = E_3,\\
F_{12} =& \alpha^2 e^{-i\alpha v} \bar{F} = -B_3,\\
F_{13} =& \alpha^2 e^{i\alpha v} F = B_2,\\
F_{23} =& -i\alpha(e^{i\alpha v} F_{\bar{\zeta}} + e^{-i\alpha v} \bar{F}_{\bar{\zeta}}) = -B_1,
\end{aligned} \tag{19}$$

where F is a function of u, ζ and $\bar{\zeta}$.

4 Conclusion

Using Maxwellian formalism, analogy between electromagnetic and gravity is not easy to understand because the equations are highly non-linear due to curvature of the space under consideration. The introduction of Debye's potential provides alternative equations, here also equations are complicated. However, the use of Newman-Penrose formalism simplifies the nature of equations. We have used this formalism to derive the equation for Debye's potential ψ in the space described by pp-waves. This in turn is useful in deriving expressions of Maxwell's electromagnetic field tensor. It is observed that the gravitation field as well as electromagnetic field described by pp-waves metric is neither purely electric nor purely magnetic type. A similar observation for the case of Vaidya metric was made earlier in [8]. For other known spacetimes of general nature, one needs to workout whether similar interaction of gravitation and electrodynamics occurs.

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