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Periodicity of Generalized Lucas Numbers and the Length of its Period Under Modulo 2^e

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Abstract

The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. In this paper, we consider the sequence associated with the sequence of classical Lucas sequence $\{L_n\}$. We consider the sequence $\{G_n\}$ defined by the recurrence relation $G_n = aL_{n-2} + bL_{n-1}$; for all $n \geq 2$; where $L_0 = 2, L_1 = 1$. We show that for every $e \geq 1$, this sequence of generalized Lucas numbers is always periodic. We prove number of results related with the periodicity of this sequence. We finally derive the length of the period of $\{G_n\}$ when considered under modulo integers 2^e ; $e \geq 1$.

Key Words: Fibonacci numbers, Lucas numbers, Generalized Lucas numbers, Periodicity of sequence.

AMS Subject Classification (2010): 11B39, 11B37, 11B50.

1. INTRODUCTION

Fibonacci sequence exhibit numerous interesting properties related to periodicity of its digits modulo 10^t. When we examine the sequence of the unit digits of the Fibonacci numbers F_n , then initially we observe no obvious pattern. But on closer look, the pattern is clearly noticed. The sequence of numbers at the unit's place of the Fibonacci numbers is repeating itself after every 60 numbers, $i.e.$ the sequence of numbers at unit's place of the Fibonacci numbers is periodic in nature. Koshy [4] used the principal of mathematical induction to prove that $F_{60n+i} \equiv F_i (mod\ 10)$; for any $i, n \geq 0$.

Let p be the smallest positive integer such that $F_{p+i} \equiv F_i\ (mod\ 10)$, for every integer i. Then we call p to be the period of the Fibonacci sequence modulo 10. Thus the period of the Fibonacci sequence modulo 10 is 60. In 1963, Geller [2] used the computers to established that $F_{n+300} \equiv F_n (mod\ 100)$ and $F_{n+1.5\times10^t} \equiv F_n (mod\ 10^t)$; where $3 \leq t \leq 6$.

In 1972, Kramer and Hoggatt Jr. [5] established the periodicity of Fibonacci numbers as well as of Lucas numbers modulo 10^n. Shah [7] obtained the identity $T_{n+12.4\times10^t} \equiv T_n(mod\ 10^t)$; where $t \geq 1$ and $n \geq 0$, for the sequence of Tribonacci numbers $\{T_n\}_{n=1}^{\infty}$ defined by $T_n = T_{n-1} + T_{n-2} + T_{n-3}$; where $T_1 = 0$ and $T_2 = T_3 = 1$.

Recently Hathiwala, Shah [3] studied the periodicity of Tetranacci numbers and proved that $t_{n+3.9\times10^r} \equiv t_n(mod\ 10^r)$, for all $r \geq 4$ and $n \geq 0$. Here we note that the Tetranacci numbers t_n are defined by the recurrence relation $t_n = t_{n-1} + t_{n-2} + t_{n-3} + t_{n-4}$; where $t_1 = t_2 = 0$ and $t_3 = t_4 = 1$.

In this paper we study the periodicity of the sequence of Generalized Lucas numbers introduced by Das, Patel, Shah [1].

Definition: If L_n is the n^{th} Lucas number, then for positive integers a, b and for all $n \geq 2$, the *Generalized Lucas sequence* $\{G_n\}$ is defined by the recurrence relation

$$G_n = aL_{n-2} + bL_{n-1}. \quad (1)$$

First few terms of this sequence are: $3a - b, -a + 2b, 2a + b, a + 3b, 3a + 4b$ … .

This paper is concerned with determining the length of the period of recurring sequence obtained by reducing the terms of sequence $\{G_n\}$ by a modulo any integer $m \geq 2$. Throughout this paper, by $G(mod\ m)$ we mean the sequence of least non-negative residues of the terms of sequence $\{G_n\}$ taken modulo $m \geq 2$. As an illustration, we consider $G(mod\ 3)$ in Table: 1.

$\boldsymbol{n}$:	$\cdots$	-2	-1	0	1	2	3	4	5
$\boldsymbol{G(mod\ 3)}$	$\cdots$	$a+2b$	$2a$	$2b$	$2a+2b$	$2a+b$	a	b	$a+b$
$\boldsymbol{n}$:	6	7	8	9	10	11	12	13	$\cdots$
$\boldsymbol{G(mod\ 3)}$	$a+2b$	$2a$	$2b$	$2a+2a$	$2a+b$	a	b	$a+b$	$\cdots$

Table:1

Here we note that $G_0 = 2b = 3a - b$, $G_1 = 2a + 2b = -a + 2b$ when considered under modulo 3. First thing which can be noticed by observing this table is that $G(mod\ 3)$ is periodic. Also it is not difficult to check that $G_{8n+i} \equiv G_i(mod\ 3)$; where $n \geq 0$. Thus the period of $G(mod\ 3)$ is 8.

We now define the length of the shortest period of sequences $\{G_n\}$ and $\{F_n\}$ modulo m.

Definition: By the length of *period* of the sequence $\{G_n\}$ modulo any positive integer m, we mean the smallest positive integer $h = h(m)$ such that $G_{h+i} \equiv G_i(mod\ m)$; for every i.

Definition: By the length of period of the sequence $\{F_n\}$ modulo any positive integer m, we mean the smallest positive integer $k = k(m)$ such that $F_{k+i} \equiv F_i(mod\ m)$; for every i.

It can be observed that values of $k(m)$ and $h(m)$ are identical for positive integer $m > 1$. The following table presents the value of $h(m)$ for $1 < m \leq 20$.

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$h(m)$	3	8	6	20	24	16	12	24	60	10	24	28	48	40	24	36	24	18	60

Table : 2

2. PRELIMINARY RESULTS

For ease we write 'h' and 'k' for h(m) and $k(m)$ respectively, when it is clear that $\{G_n\}$ and $\{F_n\}$ are considered modulo m. Following are some immediate consequences from the above definitions:

Lemma 2.1: (a) $G_{h-1} \equiv 4a - 3b (mod\ m)$ (b) $G_h \equiv 3a - b (mod\ m)$

(c) $G_{h+1} \equiv -a + 2b (mod\ m)$ (d) $G_{h+2} \equiv 2a + b (mod\ m)$

(e) $G_{n+hr} \equiv G_n (mod\ m)$; for any integer r.

Lemma 2.2: (a) $F_{k-1} \equiv 1 (mod\ m)$ (b) $F_k \equiv 0 (mod\ m)$

(c) $F_{k+1} \equiv 1 (mod\ m)$ (d) $F_{k+2} \equiv 1 (mod\ m)$

(e) $F_{n+kr} \equiv F_n (mod\ m)$; for any integer *r*.

Fact 2.3: Since $F(mod\ m)$ is always periodic, we note the following:

(i) $F_n \equiv 0 (mod\ m)$ and (ii) $F_{n+1} \equiv 1 (mod\ m)$, then $k(m) \mid n$.

The following result was proved by Wall [9] and others for the sequence $\{F_n\}$:

Theorem 2.4: $k(2^e) = 3 \times 2^{e-1}$; $e \geq 1$.

3. SOME FUNDAMENTAL RESULTS

Looking to the above discussion, the first question which arises is that 'For any given integer $m > 1$, does $G(mod\ m)$ is always periodic?' We answer this question in the following lemma. Right through the paper, we now mean 'm' to be any given positive integer greater than 1.

Lemma 3.1: The sequence $G(mod\ m)$ is always periodic.

Proof: If we consider $G(mod\ m)$, then it is clear that to be maximum only m^2 different pairs of residues are possible. Since $G(mod\ m)$ is an infinite sequence, some pair of consecutive terms of $G(mod\ m)$ must repeat. Also we know that any pair of consecutive terms of $G(mod\ m)$ completely determines the entire sequence ($i.e.$ both forward and backward), the whole sequence $G(mod\ m)$ repeats. Thus the result follows immediately.

It will always happen that the first pair to repeat will be the pair we started with, i.e. $3a - b, -a + 2b$. We now prove this result.

Lemma 3.2: The periodic sequence $G(mod\ m)$ always repeats from its starting values $3a-b, -a+2b$.

Proof: Suppose the sequence $G(mod\ m)$ does not repeat from $3a-b, -a+2b$, the first pair we started with. Then we might have the sequence of the type

$$3a-b, -a+2b, \dots, x, y, \dots, x, y, \dots ; \qquad (2)$$

where the pair $3a-b, -a+2b$ is not contained in the block $x, y, \dots, x, y$.

However, we know that this block repeats backward and forward. So the pair $3a-b, -a+2b$ cannot be in the sequence (2), which is clearly a contradiction. Hence the first pair to repeat will be the pair $3a-b, -a+2b$.

Remark: For any integer $m > 1$, when we consider $G(mod\ m)$, it is clear that $G(mod\ m)$ starts with $3a-b, -a+2b$. Thus $h(m) > 2$; for all m.

We note the following trivial fact which follows immediately from the lemma 2.1 and periodic nature of $G(mod\ m)$:

Fact 3.3: If (i) $G_n \equiv 3a-b(mod\ m)$ and (ii) $G_{n+1} \equiv -a+2b(mod\ m)$ then $h(m) \mid n$.

4. TO EXPRESS $h(m)$ IN TERMS OF m

We observe one curious feature of $G(mod\ m)$ that $h(n) \mid h(m)$, whenever $n \mid m$ This divisibility property is proved in the following theorem:

Theorem 4.1: For the sequence $\{G_n\}$, if $n \mid m$ then $h(n) \mid h(m)$.

Proof: Let $n \mid m$. Here we prove that $\{G_n\}$ repeats in the blocks of length $h = h(m)$. In fact we show that $G_{h+i} \equiv G_i(mod\ n)$; for any choice of i.

Now by lemma 2.1 (e), we have $G_{h+i} \equiv G_i(mod\ m)$. Then for some $0 \le s < m$ and for some positive integers x and y, we have

$$G_i = s + mx \text{ and } G_{h+i} = s + my. \qquad (3)$$

Also, since $n \mid m$, we assume $m = nr$, for some positive integer r. Substituting this value of m in (3) we get

$$G_i = s + nrx \text{ and } G_{h+i} = s + nry. \qquad (4)$$

Also since $n \le m$ and $0 \le s < m$, we assume that $s = s' + nz$; where z is some integer and $0 \le s' < n$.

Substituting this value of s in (4) we have $G_i = s' + n(z + rx)$ and $G_{h+i} = s' + n(z + ry)$. Considering these equations under modulo n, we now get

$$G_i \equiv s'(mod\ n) \text{ and } G_{h+i} \equiv s'(mod\ n).$$

Thus $G_{h+i} \equiv G_i(mod\ n)$ for all i, as required.

Corollary 4.2: $h(m) \mid h(mn)$; for any positive integers m and n.

Proof: Result follows immediately from above theorem, as $m \mid mn$ is always true.

Corollary 4. 3: If m is composite then $h(m)$ is also composite.

Proof: Let m be composite. Let $m = m_1 m_2$; where $1 < m_1 \leq m_2 < m$. Then by above theorem we say that $m_1 \mid m$ implies $h(m_1) \mid h(m)$. Since $h(m_1) \geq 3$; for any m_1, result follows immediately.

We try to express the period of $G(mod\, m)$ in terms of smaller parts. Let $m = \prod_{i=1}^{r} p_i^{e_i} = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$. We express $h(m)$ in terms of $h(p_i^{e_i})$.

Theorem 4. 4: $h(m) = lcm\{h(p_1^{e_1}), h(p_2^{e_2}), \dots, h(p_r^{e_r})\}$.

Proof: We have $m = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$. Then $p_i^{e_i} \mid m$; for all i. So by theorem 4.1, we have $h(p_i^{e_i}) \mid h(m)$; for all $i = 1, 2, \dots, r$.

If we denote $lcm\{h(p_1^{e_1}), h(p_2^{e_2}), \dots, h(p_r^{e_r})\}$ by L, then we have

$$L \mid h(m). \tag{5}$$

Now since $h(p_i^{e_i}) \mid L$ for all i, the sequence $G(mod\, p_i^{e_i})$ repeats in the blocks of length L; for all $i = 1, 2, \dots, r$. So by lemma 2.1 (b) and (c), we have

$$G_L \equiv 3a - b(mod\, p_i^{e_i}) \text{ and } G_{L+1} \equiv -a + 2b(mod\, p_i^{e_i}).$$

Since all $p_i^{e_i}$ are relatively primes, we have

$$G_L \equiv 3a - b(mod\, p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}) \text{ and } G_{L+1} \equiv -a + 2b(mod\, p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}).$$

Thus $G_L \equiv 3a - b(mod\, m)$ and $G_{L+1} \equiv 3a - b(mod\, m)$. Using the fact 3.3, it now follows that

$$h(m) \mid L. \tag{6}$$

Hence by (5) and (6) we get $h(m) = L$, which proves the required result.

5. TO EXPRESS $h(p^e)$ IN TERMS OF p^e

Before we develop results which speak regarding the characterization of $h(p^e)$, we look at a related result. In the following theorem we see that it is not necessary to break a modulus into its prime factorization in order to get information about $h(m)$.

Theorem 5. 1: $h([m, n]) = [h(m), h(n)]$; where $[m, n]$ denotes the least common multiple of m and n.

Proof: Since $m \mid [m, n]$ and $n \mid [m, n]$, by theorem 4.1 we have $h(m) \mid h([m, n])$ and $h(n) \mid h([m, n])$. Hence it follows that

$$[h(m), h(n)] \mid h([m, n]). \tag{7}$$

Now suppose we have the prime factorization $[m, n] = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$. So, $h([m, n]) = h(p_1^{e_1} p_2^{e_2} \dots p_r^{e_r})$.

Using theorem 4.4 we now have

$$h([m, n]) = [h(p_1^{e_1}), h(p_2^{e_2}), \dots, h(p_r^{e_r})]. \tag{8}$$

But for all i, either $p_i^{e_i} \mid m$ or $p_i^{e_i} \mid n$. So by theorem 4.1, $h(p_i^{e_i}) \mid h(m)$ or $h(p_i^{e_i}) \mid h(n)$; for all i. Thus, $h(p_i^{e_i}) \mid [h(m), h(n)]$; for all i.

So, $[h(p_1^{e_1}), h(p_2^{e_2}), \dots, h(p_t^{e_t})] \mid [h(m), h(n)]$. Using (8) we now have

$$h([m, n]) \mid [h(m), h(n)]. \tag{9}$$

Hence by (7) and (9) we finally have $h([m,n]) = [h(m), h(n)]$.

We now turn our attention in expressing $h(p^e)$ in terms of p^e for the special case $p = 2$.

Theorem 5.2: $h(2^e) = 3 \times 2^{e-1}; e \geq 1$.

Proof: To prove this result, it is first required to prove that

$$G_{3\times 2^{e-1}} \equiv G_0 = 3a - b(mod\ 2^e) \text{ and } G_{3\times 2^{e-1}+1} \equiv G_1 = -a + 2b(mod\ 2^e).$$

For $e = 1$, we get

$$G_{3\times 2^{1-1}} = G_3 = a + 3b \equiv a + b \equiv a - b \equiv 3a - b \equiv G_0(mod\ 2^1)$$

$$G_{3\times 2^{1-1}+1} = G_4 = 3a + 4b \equiv a \equiv -a + 2b \equiv G_1(mod\ 2^1).$$

Again, for $e = 2$, we get

$$G_{3\times 2^{2-1}} = G_6 = 7a + 11b \equiv 3a - b \equiv G_0(mod\ 2^2)$$

$$G_{3\times 2^{2-1}+1} = G_7 = 11a + 18b \equiv -a + 2b \equiv G_1(mod\ 2^2).$$

Thus result holds for $e = 1, 2$.

We now assume that result holds for some positive integer $t \geq 3$. i.e. let $G(2^t) = 3 \times 2^{t-1}$ holds for some integer $t \geq 3$. This implies that $G_{3\times 2^{t-1}} \equiv G_0(mod\ 2^t)$ and $G_{3\times 2^{t-1}+1} \equiv G_1(mod\ 2^t)$ holds and we prove that result is true for $e = t + 1$ also.

Now, for the sequence of Generalized Lucas numbers, Das, Patel, Shah [1] proved that $G_{2n} = (3a - b)F_{2n-1} + (-a + 2b)F_{2n}$. Also by Koshy [4], it is known that $F_{n-1}^2 + F_n^2 = F_{2n-1}$ and $F_{2n} = 2F_nF_{n-1} + F_n^2$. This gives

$$G_{2n} = (3a - b)(F_{n-1}^2 + F_n^2) + (-a + 2b)(2F_nF_{n-1} + F_n^2).$$

Taking $n = 3 \times 2^{t-1}$, we get

$$G_{3\times 2^t} = (3a - b)\left(F_{3\times 2^{t-1}-1}^2 + F_{3\times 2^{t-1}}^2\right) + 2(-a + 2b)F_{3\times 2^{t-1}}F_{3\times 2^{t-1}-1} + (-a + 2b)F_{3\times 2^{t-1}}^2.$$

Again for the sequence of Fibonacci numbers, by theorem 2.4, we have $k(2^e) = 3 \times 2^{e-1}; e \geq 1$. Thus from lemma 2.2, by taking $m = 2^t$ the following are clear:

$$\left.\begin{array}{l} \text{(i)}\ \ F_{3\times 2^{t-1}} \equiv 0\ (mod 2^t) \\ \text{(ii)}\ F_{3\times 2^{t-1}+1} \equiv 1\ (mod 2^t) \\ \text{(iii)} F_{3\times 2^{t-1}-1} \equiv 1\ (mod 2^t) \end{array}\right\} \quad (10)$$

Using (10), for some integers u and v, we now write

$G_{3\times 2^t} = (3a - b)\{(1 + 2^t u)^2 + (2^t v)^2\} + 2(-a + 2b)(2^t v)(1 + 2^t u) + (-a + 2b)(2^t v)^2.$

$\quad = (3a - b)\{1 + 2^{t+1}u + 2^{2t}u^2 + 2^{2t}v^2\} + (-a + 2b)\{2^{t+1}v + 2^{2t+1}uv + 2^{2t}v^2\}.$

$$\therefore G_{3\times 2^t} - (3a - b) = 2^{t+1}\left\{\begin{array}{c}(3a - b)u + (3a - b)2^{t-1}u^2 + (3a-b)2^{t-1}v^2 + (-a + 2b)v \\ +(-a + 2b)2^t uv + (-a + 2b)2^{t-1}v^2\end{array}\right\}.$$

$\therefore G_{3\times 2^t} \equiv 3a - b(mod\ 2^{t+1})$. This gives,

$$G_{3\times 2^t} \equiv G_0(mod\ 2^{t+1}). \quad (11)$$

Now, for the sequence of generalized Lucas numbers, Das, Patel, Shah [1] proved that

$$G_{2n+1} = (3a - b)F_{2n} + (-a + 2b)F_{2n+1}.$$

Also by Koshy [4], it is known that $F_{n-1}^2 + F_n^2 = F_{2n-1}$ and $F_{2n} = 2F_nF_{n-1} + F_n^2$. This gives $G_{2n+1} = (3a - b)(2F_nF_{n-1} + F_n^2) + (-a + 2b)(F_{n+1}^2 + F_n^2)$.

Considering $n = 3 \times 2^{t-1}$ and using (11), for some integers u, v and w, we get

$$G_{3\times 2^t+1} = (3a - b)\left(F_{3\times 2^{t-1}}^2 + 2F_{3\times 2^{t-1}}F_{3\times 2^{t-1}-1}\right) + (-a + 2b)\left(F_{3\times 2^{t-1}}^2 + F_{3\times 2^{t-1}+1}^2\right)$$

$$= (3a - b)\{(2^t u)^2 + 2(2^t u)(1 + 2^t v)\} + (-a + 2b)\{(2^t u)^2 + (1 + 2^t w)^2\}$$

$$= (3a - b)\{2^{2t}u^2 + 2^{t+1}u + 2^{2t+1}uv\} + (-a + 2b)\{2^{2t}u^2 + 1 + 2^{t+1}w + 2^{2t}w^2\}.$$

$$\therefore G_{3\times 2^t+1} - (-a + 2b)$$

$$= 2^{t+1}\left\{\begin{matrix}(3a - b)2^{t-1}u^2 + (3a - b)u + (3a - b)(2^t uv) + (-a + 2b)2^{t-1}u^2 \\ +(-a + 2b)w + (-a + 2b)2^{t-1}w^2\end{matrix}\right\}$$

Thus, we get $G_{3\times 2^t+1} \equiv (-a + 2b)\ (mod 2^{t+1})$.

This shows that

$$G_{3\times 2^t+1} \equiv G_1 (mod 2^{t+1})\ . \tag{12}$$

Using fact 3.3 and (11), (12) we can now conclude that

$$h(2^{t+1}) \mid 3 \times 2^t. \tag{13}$$

Now $2^t \mid 2^{t+1}$ implies that $h(2^t) \mid h(2^{t+1})$. Also by our induction hypothesis, we have $h(2^t) = 3 \times 2^{t-1}$. This gives

$$3 \times 2^{t-1} \mid h(2^{t+1}). \tag{14}$$

Thus using (13) and (14), we conclude that $h(2^{t+1}) = 3 \times 2^{t-1}$ or 3×2^t.

It can be observed from table: 1 that $h(4) = 6, h(8) = 12,\ h(16) = 24, \dots$. This clearly indicates that $h(2^{t+1}) \neq 3 \times 2^{t-1}$, which now gives $h(2^{t+1}) = 3 \times 2^t$. Thus result holds for $e = t + 1$.

Hence by induction, $h(2^e) = 3 \times 2^{e-1}$ is true for every positive integer e, which proves the required result.

Remark: It is clear from the theorem that the sequence $\{h(2^n)\}_{n\geq 1}$ is strictly increasing sequence.

Corollary 5.3: If n is even, then $h(n)$ is divisible by 3.

Proof: Let n be even. If we write $n = 2^r \times m$; $r \geq 1$ and m is odd, then $2^r \mid n$ implies that $h(2^r) \mid h(n)$. Thus $3 \times 2^{r-1} \mid h(n)$; $r \geq 1$, which proves the result.

Corollary 5.4: If $4 \mid n$ then $6 \mid h(n)$.

Proof: Let $4 \mid n$. If we write $n = 2^r \times m$; $r \geq 2$ and m is odd, then $2^r \mid n$ implies that $h(2^r) \mid h(n)$. Thus $3 \times 2^{r-1} \mid h(n)$; $r \geq 2$. Thus $6 \mid h(n)$, which proves the result.

Corollary 5.5: For any prime p and any even positive integer n, if $2 \mid h(p)$ then $3 \mid h(np)$.

Proof: If n is even then let $n = 2^r \times m$; r ≥ 1 and m is odd. Then, $h(np) = h(2^r \times mp)$.

If $p = 2$ then $h(np) = h(2^{r+1} \times m) = [h(2^{r+1}), k(m)] = [3 \times 2^r, h(m)]$.

Also if p is an odd prime then $h(np) = [h(2^r), h(mp)] = [3 \times 2^{r-1}, h(mp)]$.

Thus in any case $h(np)$ will always be multiple of 3; if n is even.

We also note the following trivial but interesting result:

Theorem 5.6: For any positive integers t, m, n if $t \mid h(m)$ then $t \mid h(mn)$.

This is because $h(m) \mid h(mn)$ is always true.

Here we note that the characterization of $h(p^e)$ when $p > 2$ in terms of p is extra ordinarily difficult. In fact theorem 5.2 is the best which we are able to do in connection with the value of $h(m)$. Nevertheless it definitely gives us insight into the periodic behavior of $\{G_n\}$.

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