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FIXED POINT THEOREMS AND CONTRACTIVE ITERATES

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Abstract. A few Variants of some of the generalization of Banach's Fixed Point Theorems have been established in this article.Besides some generalization results are improved here.Even in some case the conditions are relatively relaxed , while the convergence of associated sequences have been more sharp.

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1. Introduction

Essentially, Fixed point theorems provide the conditions under which maps have solution.The theory itself is a beautiful mixture of analysis(Pure and Applied), Topology and Geometry.Over the last 50 years or so the theory of fixed point has been revealed to be a very powerful and important tool in the study of non-linear phenomena. Inperticular, the techniques of the fixed point theory have been applied in various diverse fields such as Biology, Chemistry, Physics, Economics, Medicines and Game Theory etc.

It is well known and conclusively established that many existence theorems of analysis can be treated as special case of appropriate fixed point theorems. Fixed point theorems arises in various forms which depends on suitable conditions on the space and the mappings considered over the space.(Obviously the conditions must always implie that the space is non empty. Usually, the space is a topological space while barring a few situations some conditions such as continuity, compactness or atleast completeness are needed).

First the existence of a unique fixed point was given by the mathematician Banach in 1922[**2**] , which was acclaimed as Banach contraction principle which has an important role in the development of various results connected with Fixed point theory and Approximation Theory.

The Banach's fixed point theorem or the Contraction Principle concerns certain mappings of a complete metric space into itself. It lays down conditions ; sufficient for the existence and uniqueness of a fixed point. Besides, this famous classical theorem gives an iteration process through which one can obtain better approximation to the fixed point. Banach's fixed point theorem has rendered a key role in solving system of linear algebraic equations involving iteration process.

A point $x \in X$ is called a **fixed point** of the mapping $T : X \to X$ of a set X into itself if $Tx = x$.That is the image of Tx coincides with x.

Iteration procedures are used in nearly every branch of applied mathematics and convergence proof and error estimates are very often obtained by the application of Banach's fixed point theorem.

This theorem deals with contractions which are defined as follows :

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a **contraction** on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y)$$

Geometrically, this means that any point x and y have images that are closer together than those points x and y ; more precisely; the ratio $\frac{d(Tx,Ty)}{d(x,y)}$ does not exceed a constant α which is strictly less than 1.
This celebrated classical result of Banach (**Banach's Contraction Principle**) states that :
If T is a self mapping of a complete metric space X such that

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for a non negative real number $\alpha(0 < \alpha < 1)$ and for each $x, y \in X$, then T has a unique fixed point in X.
Equivalently, this can be stated as follows :
" Every contraction mapping of a complete metric space X into itself has a unique fixed point."
In particular, contraction mappings are necessarily continuous. Towards the generalization of the Banach's Contraction Principle ,$Kannan$[**1**] assumed the following condition :

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)]$$

for every $x, y \in X(x \neq y)$ and $0 < \alpha < \frac{1}{2}$ and he showed that T had a unique fixed point in X. However mappings satisfying these conditions are not necessarily continuous.
Few of the results found here refine and sharpen some of the $generalization$[**3, 4**] of Banach's Theorem.Also *Theorem;1* is an interesting result as in it condition is deals with $l^2 - space$.
Lastely it may be noted that there exist sufficient scopes for improving some of the results stated here.

2. Fixed Point Theorems and Contractive Iterates

Theorem 2.1. *Let X be a closed subspace of a Hilbert Space and $T : X \to X$ be a linear and continuous self mapping satisfying the following conditions :*

$$\begin{aligned} \|Tx - Ty\|^2 &\leq a_1^2 \frac{\|x - Ty\|^2 \|Tx - x\|^2}{[1 + \|x - y\|]^2} \\ &+ a_2^2 \frac{\|x - Ty\|^2 [1 + \|Tx - x\|]^2}{[1 + \|x - y\|]^2} \\ &+ a_3^2 \frac{\|Tx - x\|^2 [1 + \|x - Ty\|]^2}{[1 + \|x - y\|]^2} \\ &+ a_4^2 \|x - y\|^2 \end{aligned}$$

for all $x, y \in X$ with $x \neq y$,where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_2^2 + 2(a_3^2 + a_4^2) < 1$. Then T has unique fixed point in X.

Proof. Let x_0 be the arbitrary point in X and define the sequence (x_n) of iterates of T as follows :

$$x_{n+1} = Tx_n \; for \; n = 0, 1, 2, 3, ...$$

Now we proceed to show that the sequence (x_n) is a cauchy sequence.
For this observe that

$$\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\|.$$

From the given condition, we have

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &\leq a_1^2 \frac{\|x_n - Tx_{n-1}\|^2 \|Tx_n - x_n\|^2}{[1 + \|x_n - x_{n-1}\|]^2} \\ &+ a_2^2 \frac{\|x_n - Tx_{n-1}\|^2 [1 + \|Tx_n - x_n\|]^2}{[1 + \|x_n - x_{n-1}\|]^2} \\ &+ a_3^2 \frac{\|Tx_n - x_n\|^2 [1 + \|x_n - Tx_{n-1}\|]^2}{[1 + \|x_n - x_{n-1}\|]^2} \\ &+ a_4^2 \|x_n - x_{n-1}\|^2\end{aligned}$$

This gives,

$$[(1 - a_3^2) + (2 + \|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|]\|x_{n+1} - x_n\|^2$$
$$\leq [a_4^2 (1 + \|x_n - x_{n-1}\|)^2]\|x_n - x_{n-1}\|^2$$

and so

$$\|x_{n+1} - x_n\|^2 \leq p(n)\|x_n - x_{n-1}\|^2$$

where

$$p(n) = \frac{a_4^2 (1 + \|x_n - x_{n-1}\|)^2}{(1 - a_3^2) + (2 + \|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|} forn = 1, 2, 3, ...$$

Clearly

$$p(n) < 1, \forall n \geq 1 \ as \ a_2^2 + 2(a_3^2 + a_4^2) < 1$$

Repeating the above procedure, we find some $s < 1$ such that

$$\|x_{n+1} - x_n\|^2 \leq s^n \|x_1 - x_0\|^2$$

consequently,

$$\|x_{n+1} - x_n\|^2 \to o \quad as \quad n \to 0.$$

Hence, (x_n) is a cauchy sequence in X. But X is a closed subset of a Hilbert space and so by the completeness of X, there exists $\mu \in X$ which is the limit point of the sequence (x_n),that is

$$\lim_{n \to \infty} x_n = \mu$$

Also,

$$(x_{n+1}) = (Tx_n)$$

is a subsequence of (x_n) and hence convergese to the same limit μ.

Since T is continuous,we get

$$\begin{aligned} T(\mu) &= T(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} Tx_n \\ &= \lim_{n\to\infty} x_{n+1} \\ &= \mu \end{aligned}$$

Hence T has a fixed point μ in X.
Next to show the uniqueness of the fixed point, let us take $\nu(\mu \neq \nu)$ to be another fixed point of T; that is

$$T\nu = \nu$$

while

$$\|\mu - \nu\| \neq 0$$

Hence it follows that

$$\begin{aligned} \|\mu - \nu\|^2 &= \|T\mu - T\nu\| \\ &\leq a_1^2 \frac{\|\mu - T\nu\|^2 \|T\mu - \mu\|^2}{[1 + \|\mu - \nu\|]^2} \\ &+ a_2^2 \frac{\|\mu - T\nu\|^2 [1 + \|T\mu - \mu\|]^2}{[1 + \|\mu - \nu\|]^2} \\ &+ a_3^2 \frac{\|T\mu - \mu\|^2 [1 + \|\mu - T\nu\|]^2}{[1 + \|\mu - \nu\|]^2} \\ &+ a_4^2 \|\mu - \nu\|^2 \end{aligned}$$

$$\therefore \|\mu - \nu\|^2 < (a_2^2 + a_4^2)\|\mu - \nu\|^2$$

which is a contradiction as

$$\begin{aligned} a_2^2 + a_4^2 &< a_2^2 + 2 * (a_3^2 + a_4^2) \\ &< 1 \end{aligned}$$

Hence $\mu = \nu$.

Thus, T has a unique fixed point μ in X.

□

Theorem 2.2. *Let X be a closed subspace of a Hilbert Space and $T : X \to X$ be a linear and continuous self mapping satisfying the following conditions :*

$$\begin{aligned} \|Tx - Ty\|^p &\leq a_1 \frac{\|x - Ty\|^p [1 + \|y - Tx\|^p]}{[1 + \|Tx - Ty\|^p]} \\ &+ a_2 \frac{\|y - Tx\|^p [1 + \|x - Ty\|^p]}{[1 + \|Tx - Ty\|^p]} \\ &+ a_3 [\|x - Ty\|^p + \|y - Tx\|^p] \\ &+ a_4 \|x - y\|^p \end{aligned}$$

for all $x, y \in X$ with $x \neq y$ and $p \in \mathbb{N}$,where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1 + 2^{p+1}(a_2 + a_3) + a_4 < 1$.Then T has unique fixed point in X.

Proof. Let x_0 be the arbitrary point in X and define the sequence (x_n) of iterates of T as follows :

$$x_{n+1} = Tx_n \ for \ n = 0, 1, 2, 3, ...$$

Now we proceed to show that the sequence (x_n) is a cauchy sequence.
For this observe that

$$\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\|.$$

From the given condition, we have

$$\begin{aligned}
\|x_{n+1} - x_n\|^p &\leq a_1 \frac{\|x_n - Tx_{n-1}\|^p[1 + \|x_{n-1} - Tx_n\|^p]}{[1 + \|Tx_n - Tx_{n-1}\|^p]} \\
&+ a_2 \frac{\|x_{n-1} - Tx_n\|^p[1 + \|x_n - Tx_{n-1}\|^p]}{[1 + \|Tx_n - Tx_{n-1}\|^p]} \\
&+ a_3[\|x_n - Tx_{n-1}\|^p + \|x_{n-1} - Tx_n\|^p] \\
&+ a_4\|x_n - x_{n-1}\|^p
\end{aligned}$$

This gives,

$$\begin{aligned}
& [(1 - 2^p a_2 - 2^p a_3) + (1 - 2^p a_3)\|x_{n+1} - x_n\|^p]\|x_{n+1} - x_n\|^p \\
\leq \ & [(2^p a_2 + 2^p a_3 + a_4) + (2^p a_3 + a_4)\|x_{n+1} - x_n\|^p]\|x_n - x_{n-1}\|^p
\end{aligned}$$

and so

$$\|x_{n+1} - x_n\|^p \leq p(n)\|x_n - x_{n-1}\|^p$$

where

$$p(n) = \frac{(2^p a_2 + 2^p a_3 + a_4) + (2^p a_3 + a_4)\|x_{n+1} - x_n\|^p}{(1 - 2^p a_2 - 2^p a_3) + (1 - 2^p a_3)\|x_{n+1} - x_n\|^p} for n = 1, 2, 3, ...$$

Clearly

$$p(n) < 1, \forall n \geq 1 \ as \ a_1 + 2^{p+1}(a_2 + a_3) + a_4 < 1$$

Repeating the above procedure, we find some $s < 1$ such that

$$\|x_{n+1} - x_n\|^p \leq s^n\|x_1 - x_0\|^p$$

consequently,

$$\|x_{n+1} - x_n\| \to o \quad as \quad n \to 0.$$

Hence, (x_n) is a cauchy sequence in X. But X is a closed subset of a Hilbert space and so by the completeness of X, there exists $\mu \in X$ which is the limit point of the sequence (x_n),that is

$$\lim_{n\to\infty} x_n = \mu$$

Also,

$$(x_{n+1}) = (Tx_n)$$

is a subsequence of (x_n) and hence converges to the same limit μ.
Since T is continuous,we get

$$\begin{aligned}
T(\mu) &= T(\lim_{n\to\infty} x_n) \\
&= \lim_{n\to\infty} Tx_n \\
&= \lim_{n\to\infty} x_{n+1} \\
&= \mu
\end{aligned}$$

Hence T has a fixed point μ in X.

Next to show the uniqueness of the fixed point, let us take $\nu(\mu \neq \nu)$ to be another fixed point of T; that is

$$T\nu = \nu$$

while

$$\|\mu - \nu\| \neq 0$$

Hence it follows that

$$\begin{aligned} \|\mu - \nu\|^p &= \|T\mu - T\nu\|^p \\ &\leq a_1 \frac{\|\mu - T\nu\|^p[1 + \|\nu - T\mu\|^p]}{[1 + \|T\mu - T\nu\|^p]} \\ &+ a_2 \frac{\|\nu - T\mu\|^p[1 + \|\mu - T\nu\|^p]}{[1 + \|T\mu - T\nu\|^p]} \\ &+ a_3[\|\mu - T\nu\|^p + \|\nu - T\mu\|^p] \\ &+ a_4\|\mu - \nu\|^p \end{aligned}$$

$$\therefore \|\mu - \nu\|^p < (a_1 + a_2 + 2a_3 + a_4)\|\mu - \nu\|^p$$

which is a contradiction as

$$\begin{aligned} a_1 + a_2 + 2a_3 + a_4 &< a_1 + 2^{p+1}(a_2 + a_3) + a_4 \\ &< 1 \end{aligned}$$

Hence $\mu = \nu$.

Thus, T has a unique fixed point μ in X.

□

Similar is the proof of the following result :

Theorem 2.3. *Let X be a closed subspace of a Hilbert Space and $T : X \to X$ be linear and continuous self mapping satisfying the following conditions :*

$$\begin{aligned} \|Tx - Ty\|^p &\leq a_1 \frac{\|x - y\|^p[1 + \|y - Ty\|^p]}{[1 + \|x - y\|^p]} \\ &+ a_2[\|x - Tx\|^p + \|y - Ty\|^p] \\ &+ a_3[\|x - Ty\|^p + \|y - Tx\|^p] \\ &+ a_4\|x - y\|^p \end{aligned}$$

for all $x, y \in X$ with $x \neq y$ and $p \in \mathbb{N}$, where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1 + 2a_2 + 2^{p+1}a_3 + a_4 < 1$. Then T has unique fixed point in X.

Theorem 2.4. *Let X be a closed subspace of a Hilbert Space and $T : X \to X$ be a linear and continuous self mapping satisfying the following conditions :*

$$\begin{aligned} \|Tx - Ty\|^p &\leq a_1 \frac{\|x-y\|^p[1+\|x-Tx\|^p]}{[1+\|y-Ty\|^p]} \\ &+ a_2 \frac{\|x-Tx\|^p[1+\|x-y\|^p]}{[1+\|y-Ty\|^p]} \\ &+ a_3[\|x-Tx\|^p + \|x-y\|^p] \\ &+ a_4\|x-y\|^p \end{aligned}$$

for all $x, y \in X$ with $x \neq y$ and $p \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$,where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1 + a_2 + 2a_3 + a_4 < 1$.Then T has unique fixed point in X.

Proof. Let x_0 be the arbitrary point in X and define the sequence (x_n) of iterates of T as follows :

$$x_{n+1} = Tx_n \ for \ n = 0, 1, 2, 3, ...$$

Now we proceed to show that the sequence (x_n) is a cauchy sequence. For this observe that

$$\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\|.$$

From the given condition, we have

$$\begin{aligned} \|x_{n+1} - x_n\|^p &\leq a_1 \frac{\|x_n - x_{n-1}\|^p[1+\|x_n - Tx_n\|^p]}{[1+\|x_{n-1} - Tx_{n-1}\|^p]} \\ &+ a_2 \frac{\|x_n - Tx_n\|^p[1+\|x_n - x_{n-1}\|^p]}{[1+\|x_{n-1} - Tx_{n-1}\|^p]} \\ &+ a_3[\|x_n - Tx_n\|^p + \|x_n - x_{n-1}\|^p] \\ &+ a_4\|x_n - x_{n-1}\|^p \end{aligned}$$

This gives,

$$[(1-a_2-a_3) + (1-a_1-a_2-a_3)\|x_n - x_{n-1}\|^p]\|x_{n+1} - x_n\|^p$$
$$\leq [(a_1+a_3+a_4) + (a_3+a_4)\|x_n - x_{n-1}\|^p]\|x_n - x_{n-1}\|^p$$

and so

$$\|x_{n+1} - x_n\|^p \leq p(n)\|x_n - x_{n-1}\|^p$$

where

$$p(n) = \frac{(a_1+a_3+a_4) + (a_3+a_4)\|x_n - x_{n-1}\|^p}{(1-a_2-a_3) + (1-a_1-a_2-a_3)\|x_n - x_{n-1}\|^p} forn = 1, 2, 3, ...$$

Clearly

$$p(n) < 1, \forall n \geq 1 as a_1 + a_2 + 2a_3 + a_4 < 1$$

Repeating the above procedure, we find some $s < 1$ such that

$$\|x_{n+1} - x_n\|^p \leq s^n\|x_1 - x_0\|^p$$

consequently,

$$\|x_{n+1} - x_n\| \to o \quad as \quad n \to 0.$$

Hence, (x_n) is a cauchy sequence in X. But X is a closed subset of a Hilbert space and so by the completeness of X, there exists $\mu \in X$ which is the limit point of the sequence (x_n),that is

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Also,

$$(x_{n+1}) = (Tx_n)$$

is a subsequence of (x_n) and hence convergese to the same limit μ. Since T is continuous,we get

$$\begin{aligned} T(\mu) &= T(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} Tx_n \\ &= \lim_{n\to\infty} x_{n+1} \\ &= \mu \end{aligned}$$

Hence T has a fixed point μ in X.

Next to show the uniqueness of the fixed point, let us take $\nu(\mu \neq \nu)$ to be another fixed point of T; that is

$$T\nu = \nu$$

while

$$\|\mu - \nu\| \neq 0$$

Hence it follows that

$$\begin{aligned} \|\mu - \nu\|^p &= \|T\mu - T\nu\| \\ &\leq a_1 \frac{\|\mu - \nu\|^p[1 + \|\mu - T\mu\|^p]}{[1 + \|\nu - T\nu\|^p]} \\ &+ a_2 \frac{\|\mu - T\mu\|^p[1 + \|\mu - \nu\|^p]}{[1 + \|\nu - T\nu\|^p]} \\ &+ a_3[\|\mu - T\mu\|^p + \|\mu - \nu\|^p] \\ &+ a_4\|\mu - \nu\|^p \end{aligned}$$

$$\therefore \|\mu - \nu\|^p < (a_1 + a_3 + a_4)\|\mu - \nu\|^p$$

which is a contradiction as

$$\begin{aligned} a_1 + a_3 + a_4 &< a_1 + a_2 + 2a_3 + a_4 \\ &< 1 \end{aligned}$$

Hence $\mu = \nu$. Thus, T has a unique fixed point μ in X. □

In an identical way one can derive the following :

Theorem 2.5. *Let X be a closed subspace of a Hilbert Space and $T : X \to X$ be a linear and continuous self mapping satisfying the following conditions :*

$$\begin{aligned}\|Tx - Ty\|^p &\leq a_1 \frac{\|x - Tx\|^p[1 + \|y - Ty\|^p]}{[1 + \|x - y\|^p]} \\ &+ a_2 \frac{\|y - Ty\|^p[1 + \|x - Tx\|^p]}{[1 + \|x - y\|^p]} \\ &+ a_3[\|x - Tx\|^p + \|y - Ty\|^p] \\ &+ a_4[\|y - Ty\|^p + \|x - Tx\|^p]\end{aligned}$$

for all $x, y \in X$ with $x \neq y$ and $p \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1 + a_2 + 2(a_3 + a_4) < 1$. Then T has unique fixed point in X.

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