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Weak Domination Number of Corona Graphs

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Abstract

Let $G = (V, E)$ be a graph and $uv \in E$ be an edge then v weakly dominates u if $deg(v) \leq deg(u)$. A set S is a *weak dominating set* ($wd - set$) if every vertex $u \in V - S$ is weakly dominated by some v in S. The weak domination number $\gamma_w(G)$ of G is the minimum cardinality of a weak dominating set. We investigate weak domination number of some corona related graphs.

Keywords: Dominating set, Domination number, Weak dominating set, Weak domination number, Corona graph.

AMS Subject classification number: 05C69, 05C76

1 Introduction

We consider simple, finite, connected and undirected graph G with vertex set V and edge set E. For all standard terminology and notations we follow West [19] while the terms related to the theory of domination in graphs are used in the sense of Haynes *et al.* [9].

Definition 1.1. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is a *minimal dominating set* if no proper subset $S' \subset S$ is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in graph G.

The number of edges incident to a vertex v is called degree of v denoted as $deg(v)$. The maximum and minimum degrees of vertices in V are denoted by $\triangle(G)$ and $\delta(G)$, respectively.

Definition 1.2. For any graph G and $uv \in E(G)$. Then, v weakly dominates u (u strongly dominates v) if $deg(v) \leq deg(u)$. A set S is a *weak dominating set* (*strong dominating set*) if every vertex $u \in V - S$ is weakly (strongly) dominated by some v in S. Weak dominating set and strong dominating set are abbreviated as $wd - set$ and $sd - set$ respectively. The *weak domination number* $\gamma_w(G)$ of G is the minimum cardinality of a weak dominating set. Similarly, we can define the *strong domination number* $\gamma_{st}(G)$.

The concept of weak (strong) domination was introduced by Sampathkumar and Pushpa Latha [13]. The following practical situation of weak and strong domination provided the reason for the initiation of this concepts in graph theory.

Consider a network of roads connecting a number of locations. In such a network, the degree of a vertex v is the number of roads meeting at v. Suppose $deg(v) \leq deg(u)$. Naturally, the traffic at v is lighter than that at u. If we consider the traffic between u and v, preference should be given to the vehicles going from u to v. Thus, in some sense, v weakly dominates u and u strongly dominates v.

Some bounds on $\gamma_{st}(G)$ were investigated by Rautenbach [10, 11]. Also some bounds on strong and weak domination numbers were investigated by Sampathkumar and Pushpa Latha [13] while Bhat *et al.* [1] have improved these bounds and reported the graphs achiveing such bounds. Rautenbach and Zverovich [12] have studied results on the NP-completeness related to strong dominating set and weak dominating set. Desai and Gangadharappa [3] have investigated upper bounds on strong domination number for trees. Gani and Ahamed [5] have introduced the concept of strong and weak domination in fuzzy graphs and also provided some examples to explain various notions. Vaidya and Karkar [16, 17] have investigated the strong domination number of some path related graphs as well as independent strong domination number of the graph obtained by switching of a vertex in path. In the same paper they have explored the concept of d-balanced graph. The strong domination number of some wheel related graphs are studied by Vaidya and Mehta [18] while the relations between strong domination and weak domination number are reported in Boutrig and Chellali [2]. Some relations between strong domination and maximum degree of the graph as well as weak domination and minimum degree of the graph are established by Swaminathan and Thangaraju [15].

2 Main Results

In this section we have obtained weak domination number of corona, edge corona and neighborhood corona graphs. The concept of corona of two graphs was introduced by Frucht and Harary in [4].

Definition 2.1. Let G and H be two graphs on n and m vertices, respectively. The *corona* $G \circ H$ of G and H is defined as the graph obtained by taking one copy of G and n copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.

Proposition 2.2. [6] Let G be a connected graph of order n and let H be any graph of order m. Then, $\gamma(G \circ H) = n$.

Theorem 2.3. Let G be a connected graph of order n and let H be any simple graph of order m. Then, $\gamma_w(G \circ H) = n\gamma_w(H)$.

Proof: Let $V(G) = \{v_1, v_2, \dots v_n\}$ and $V(\mathrm{H}) = \{u_1, u_2, \dots u_m\}$. In $G \circ H$, let's denote the vertices of i^{th} copy of the graph H by $u_1^i, u_2^i, \dots u_m^i$ for $1 \leq i \leq n$, that is, $V(H_i) = \{u_1^i, u_2^i, \dots u_m^i\}$, $1 \leq i \leq n$. By the definition of the corona graph, it is very clear that $d_{G \circ H}(u_j^i) < d_{G \circ H}(v_i)$, $1 \leq i \leq n$ and $1 \leq j \leq m$. Therefore, every vertex u_j^i can weakly dominate the vertex v_i from G, other than some vertices of i^{th} copy of H. Thus, it is enough to consider $\gamma_w(H)$ number of vertices from each copy of the graph H to weakly dominate all the vertices of H. As there are n copies of H in $G \circ H$, $n\gamma_w(H)$ number of vertices are enough to consider in weak dominating set for $G \circ H$. Hence, $\gamma_w(G \circ H) = n\gamma_w(H)$.

Illustration 2.4. In Figure 1, $S = \{u_1^1, u_1^2, u_1^3, u_1^4, u_1^5\}$ is weak dominating set of the graph $C_5 \circ P_2$ and $\gamma_w(C_5 \circ P_2) = 5$.

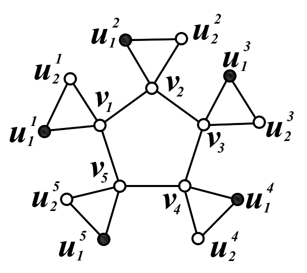

Figure 1. $C_5 \circ P_2$

Definition 2.5. Duplication of a vertex v_k by a new edge $e = v'_k v''_k$ in a graph G produces a new graph G such that $N(v'_k) = \{v_k, v''_k\}$ and $N(v''_k) = \{v_k, v'_k\}$.

Corollary 2.6. Let G be any graph with n vertices and G' be a graph obtained by duplication of every vertex of a connected graph G by an edge then $\gamma_w(G') = n$.

Proof: In $G \circ H$ let G be any connected graph with n vertices and $H = P_2$ then $G' \cong G \circ P_2$. Therefore, $\gamma_w(G') = \gamma_w(G \circ P_2)$. We know that $\gamma_w(P_2) = 1$ so, by Theorem 2.3, $\gamma_w(G \circ P_2) = n\gamma_w(P_2) = n$. Hence, $\gamma_w(G') = n$.

Definition 2.7. The helm H_n is a graph obtained from wheel by attaching a pendant edge to each rim vertex. It contains three types of vertices, the vertex of degree n called apex, n pendant vertices and n vertices of degree four.

Corollary 2.8. $\gamma_w(H_n) = n$.

Proof: In $G \circ H$ let G be a cycle with n vertices and $H = K_1$ then $C_n \circ K_1 \cong H_n$. Therefore, $\gamma_w(H_n) = \gamma_w(C_n \circ K_1)$. We know that $\gamma_w(K_1) = 1$ so, by Theorem 2.3, $\gamma_w(C_n \circ K_1) = n\gamma_w(K_1) = n$. Hence, $\gamma_w(H_n) = n$.

The concept of edge corona of two graphs was introduced by Hou and Shiu [8] which is defined as follows.

Definition 2.9. Let G and H be two graphs on n and m vertices, k and l edges, respectively. The edge corona $G \diamond H$ of G and H is defined as the graph obtained by taking one copy of G and k copies of H, and then joining two end vertices of the i-th edge of G to every vertex in the i-th copy of H.

Theorem 2.10. Let G be a connected graph of order n and size k and let H be any simple graph of order m. Then, $\gamma_w(G \diamond H) = k\gamma_w(H)$.

Proof: Let $V(G) = \{v_1, v_2, \dots v_n\}$, $E(G) = \{e_1, e_2, \dots e_k\}$ and $V(\mathrm{H}) = \{u_1, u_2, \dots u_m\}$. In $G \diamond H$, let's denote the vertices of i^{th} copy of the graph H by $u^i_1, u^i_2, \dots u^i_m$ for $1 \leq i \leq k$, that is, $V(H_i) = \{u^i_1, u^i_2, \dots u^i_m\}$, $1 \leq i \leq k$. By the definition of the edge corona graph, it is very clear that $d_{G \diamond H}(u^i_j) \leq d_{G \diamond H}(v_i)$, $1 \leq i \leq k$ and $1 \leq j \leq m$. Therefore, every vertex u^i_j can weakly dominate two vertices v_i and v_{i+1} from G other than some vertices of i^{th} copy of H and to weakly dominate all the vertices of i^{th} copy H we need $\gamma_w(H)$ number of vertices from i^{th} copy of H. Now as there are k edges in G, by the definition of edge corona graph there are k copies of H in $G \diamond H$. Therefore, $k\gamma_w(H)$ number of vertices are enough to weakly dominate all the vertices of the graph $G \diamond H$. Hence, $\gamma_w(G \diamond H) = k\gamma_w(H)$.

Illustration 2.11. In Figure 2, $S = \{u^1_1, u^2_1, u^3_1\}$ is weak dominating set of the graph $P_4 \diamond P_2$ and $\gamma_w(P_4 \diamond P_2) = 3$.

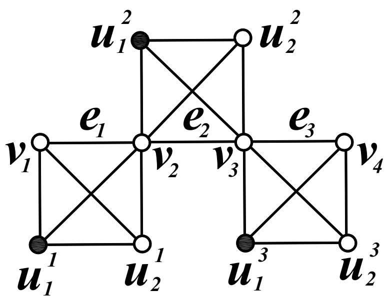

Figure 2. $P_4 \diamond P_2$

Definition 2.12. Duplication of an edge $e = uv$ by a new vertex v' in a graph G produces a new graph G' by adding a vertex v' such that $N(v') = \{u, v\}$.

Corollary 2.13. Let G' be the graph obtained by duplication of each edge of P_n by a vertex then $\gamma_w(G') = n - 1$.

Proof: In graph $P_n \diamond H$ if we consider $H = K_1$ then $G' \cong P_n \circ K_1$. Therefore, $\gamma_w(G') = \gamma_w(P_n \diamond K_1)$. As there are $n-1$ edges in $P_n \diamond K_1$ and $\gamma_w(K_1) = 1$, by the Theorem 2.10, $\gamma_w(P_n \diamond K_1) = (n-1)\gamma(K_1) = n - 1$. Hence, $\gamma_w(G') = n - 1$.

Definition 2.14. The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

Corollary 2.15. $\gamma_w(M(C_n)) = n$.

Proof: Consider $H = K_1$ in graph $C_n \diamond H$ then $C_n \diamond K_1 \cong M(C_n)$. Therefore, $\gamma_w(M(C_n)) = \gamma_w(C_n \diamond K_1)$. As there are n edges in $C_n \diamond K_1$ and $\gamma_w(K_1) = 1$ by Theorem 2.10, $\gamma_w(C_n \diamond K_1) = n\gamma_w(K_1) = n$. Hence, $\gamma_w(M(C_n))$.

Definition 2.16. A *friendship graph* F_n is a one point union of n copies of cycle C_3.

Corollary 2.17. $\gamma_w(F_n) = n$.

Proof: In $K_{1,n} \diamond H$ if we consider $H = K_1$ then $K_{1,n} \diamond K_1 \cong F_n$. Therefore, $\gamma_w(K_{1,n} \diamond K_1) = \gamma_w(F_n)$. As there are n number of edges in $K_{1,n} \diamond K_1$ and $\gamma_w(K_1) = 1$, by the Theorem 2.10, $\gamma_w(K_{1,n} \diamond K_1) = n\gamma_w(K_1) = n$. Hence, $\gamma_w(F_n) = n$.

Illustration 2.18. In Figure 3, $S = \{u_1^1, u_1^2, u_1^3\}$ is weak dominating set of the graph $K_{1,3} \diamond K_1$ and $\gamma_w(K_{1,3} \diamond K_1) = 3$.

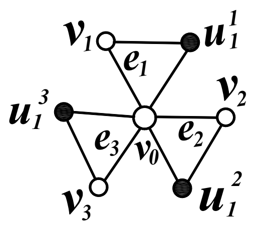

Figure 3. $K_{1,3} \diamond K_1$

Gopalapillai [7] has introduced the another variant of the corona operation recently.

Definition 2.19. Let G and H be two graphs on n and m vertices. Then the *neighborhood corona*, $G \bigstar H$ is the graph obtained by taking one copy of G and n copies of H, and then joining each neighbor of i^{th} vertex of G to every vertex in the i^{th} copy of H.

Theorem 2.20. $\gamma_w(P_n \bigstar G) = n\gamma_w(G)$.

Proof: Let $V(P_n) = \{v_1, v_2, \ldots v_n\}$ and $V(\mathrm{G}) = \{u_1, u_2, \ldots u_m\}$. In $P_n \bigstar G$, let us denote the vertices of i^{th} copy of the graph G by $u_1^i, u_2^i, \ldots u_m^i$ for $1 \leq i \leq n$, and $M_i = \{u_1^i, u_2^i, \ldots u_m^i\}$, $1 \leq i \leq n$. It is very clear that $d_{P_n \bigstar G}(v_i) = (m+1)d_{P_n}(v_i)$, for $1 \leq i \leq n$ and $d_{P_n \bigstar G}(u_j^i) = d_{P_n}(v_i) + d_G(u_j)$, for $1 \leq i \leq n$ and $1 \leq j \leq m$. Hence, $d_{P_n \bigstar G}(u_j^i) \leq d_{P_n \bigstar G}(v_i)$. Therefore, every vertex u_j^i can weakly dominate its adjacent vertices of the path P_n from the graph $P_n \bigstar G$ also. Now to weakly dominate all the vertices of the graph G we need $\gamma_w(G)$ number of vertices from the graph G. As there are n copies of the graph G in the graph $P_n \bigstar G$ it is enough to consider $n\gamma_w(G)$ number of vertices in weak dominating set and these vertices also weakly dominate all the vertices of the path P_n from the graph $P_n \bigstar G$. Therefore, these $n\gamma_w(G)$ number of vertices forms a weak dominating set of minimum cardinality. Hence, $\gamma_{st}(P_n \bigstar G) = n\gamma_w(G)$.

Illustration 2.21. In Figure 4, $S = \{u_1^1, u_1^2, u_1^3, u_1^4\}$ is weak dominating set of the graph $P_4 \bigstar P_2$ and $\gamma_w(P_4 \bigstar P_2) = 4$.

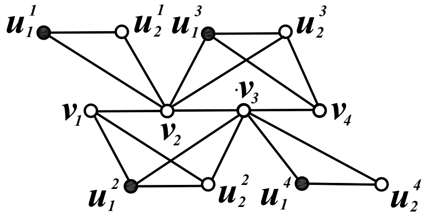

Figure 4. $P_4 \bigstar P_2$

In neighborhood corona $G \bigstar H$ if we consider $H = K_1$ then $G \bigstar H$ becomes a splitting graph. The splitting graph is introduced by Sampathkumar and Walikar [14].

Definition 2.22. For a graph G the *splitting graph* $S^{'}(G)$ of a graph G is obtained by adding a new vertex $v^{'}$ corresponding to each vertex v of G such that $N(v) = N(v^{'})$.

Corollary 2.23. For $n \geq 4$, $\gamma_w(S'(P_n)) = n$.

Proof: In the graph $P_n \bigstar G$ if we consider $G = K_1$ then $P_n \bigstar K_1 \cong S'(P_n)$. Therefore, $\gamma_w(S'(P_n)) = \gamma_w(P_n \bigstar K_1)$. We know that $\gamma_w(K_1) = 1$ Therefore, by the Theorem 2.20, $\gamma_w(P_n \bigstar K_1) = \gamma_w(S'(P_n)) = n$. Hence, $\gamma_w(S'(P_n)) = n$.

Corollary 2.24. For $\gamma_w(C_n \bigstar G) = n\gamma_w(G)$.

Proof: In Theorem 2.20 if we replace P_n by C_n then by the same argument given in Theorem 2.20 we can observe that $d_{C_n \bigstar G}(u_j^i) \leq d_{C_n \bigstar G}(v_i)$ and it is enough to consider $n\gamma_w(G)$ number of vertices from the graph G to form a weak dominating set of minimum cardinality for $C_n \bigstar G$. Hence, $\gamma_w(C_n \bigstar G) = n\gamma_w(G)$.

Corollary 2.25. $\gamma_w(S'(C_n)) = n$.

Proof: In the graph $C_n \bigstar G$ if we consider $G = K_1$ then $C_n \bigstar K_1 \cong S'(C_n)$. Therefore, $\gamma_w(S'(C_n)) = \gamma_w(C_n \bigstar K_1)$. We know that $\gamma_w(K_1) = 1$. Therefore, by the Corollary 2.24, $\gamma_w(C_n \bigstar K_1) = \gamma_w(S'(C_n)) = n$. Hence, $\gamma_w(S'(C_n)) = n$.

3 Conclusions

The concept of weak domination in graphs relates dominating sets and degree of vertices. The weak domination number of some standard graphs are already available in the literature while we have investigated the weak domination number for the larger graphs obtained by corona operations. To derive similar results for other graph families as well as in the context of various domination models are open areas of research.

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