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BSM OPTION PRICING FORMULAS THROUGH PROBABILISTIC APPROACH

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ABSTRACT. The Black-Schole-Merton (BSM) formula for European option for the plain vanilla payoff function is derived using probabilistic approach in [5]. In this paper, the BSM formulas for an European option for the standard power payoff function as well as for the exponential payoff function using probabilistic approach are derived. The BSM formula remains to be verified for the 'log' and 'modified log' payoff functions.

Keywords: European BSM option pricing formula, standard power and exponential options.
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1. INTRODUCTION

The BSM formulas of European option for various payoff functions have been derived by solving BSM differential equation (see [1]). However, so far, the BSM formula of European option has been derived through probabilistic approach only for the plain vanilla payoff function (see [5]). Main characteristic of probabilistic approach is that this method does not use the BSM differential equation. Let S_T be the stock price at the expiration time T and K be the strike price. Then the standard power payoff function is $\max\{S_T^p - K, 0\}$, where $p > 0$, and the exponential payoff function is $\max\{e^{S_T} - e^K, 0\}$. In this paper, the BSM formulas for the European call option for these two payoff functions through probabilistic approach are derived. These formulas and the formulas derived using BSM differential equation are same.

2. BSM FORMULA FOR STANDARD POWER OPTION

Let S_0 denote the stock price at time 0, let r denote the risk-free interest rate, let σ denote the volatility of stock price, let S_T denote the stock price at expiration time T, and let K be the strike price. Let x be the continuously compounded rate of return on the stock price S_0 realized between time 0 to T. Then we should have $S_T = S_0 e^{xT}$. So we have $x = \frac{1}{T}\ln(\frac{S_T}{S_0})$. Using lognormal property of stock price, we can show that x follows $\varphi(r - \frac{\sigma^2}{2}, \frac{\sigma^2}{T})$, where $\varphi(\mu, \eta^2)$ denotes the normal distribution with mean μ and standard deviation η. Hence $S_T = S_0 e^{xT}$ can be expressed as

$$S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z},$$

where Z is a standard normal random variable. Now let $p > 0$. Then, by above arguments, we can show that

$$S_T^p = S_0^p e^{p(r-\frac{\sigma^2}{2})T+p\sigma\sqrt{T}Z}. \tag{2.1}$$

Let I be the indicator random variable for the standard power option. Then

$$I = \begin{cases} 1, & \text{if } S_T^p > K \\ 0, & \text{if } S_T^p \le K. \end{cases} \tag{2.2}$$

Lemma 2.1. *Let Z and I be as in Equations* (2.1)*and* (2.2)*, respectively. Then*

$$I = \begin{cases} 1, & \textit{if } Z > p\sigma\sqrt{T} - d_1 \\ 0, & \textit{otherwise} \end{cases} \tag{2.3}$$

where $d_1 = \frac{\ln\left(\frac{S_0}{K^{1/p}}\right)+(r+(2p-1)\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$.

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Proof. Note that

$$\begin{aligned} S_T^p > K &\Leftrightarrow S_0^p e^{p(r-\frac{\sigma^2}{2})T + p\sigma\sqrt{T}Z} > K \\ &\Leftrightarrow Z > \frac{\ln\left(\frac{K^{\frac{1}{p}}}{S_0}\right) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ &\Leftrightarrow Z > p\sigma\sqrt{T} - d_1. \end{aligned}$$

This completes the proof. □

Lemma 2.2. *The expected value $E[I] = N(d_1 - p\sigma\sqrt{T})$, where $N(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{x^2}{2}}dx$ is the cumulative probability distribution function for a standard normal random variable.*

Proof. The expected value $E(I)$ of the indicator random variable I is

$$\begin{aligned} E[I] &= P(S_T^p > K) \\ &= P(Z > p\sigma\sqrt{T} - d_1) \ \ \text{(by Lemma 2.1)} \\ &= P(Z < d_1 - p\sigma\sqrt{T}) \ \ (\because P(Z > \lambda) = P(Z < -\lambda)) \\ &= N(d_1 - p\sigma\sqrt{T}). \end{aligned}$$

This proves the result. □

Lemma 2.3. *The expected value $E[IS_T^p]$ of the random variable IS_T^p is*

$$E[IS_T^p] = S_0^p e^{p(r-\frac{\sigma^2}{2})T + \frac{1}{2}\sigma^2 p^2 T} N(d_1).$$

Proof. Taking $c = p\sigma\sqrt{T} - d_1$, it follows from Lemma 2.2 that

$$\begin{aligned} E[IS_T^p] &= \int_c^{\infty} S_0^p e^{p(r-\frac{\sigma^2}{2})T + p\sigma\sqrt{T}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \frac{S_0^p e^{p(r-\frac{\sigma^2}{2})T + \frac{1}{2}\sigma^2 p^2 T}}{\sqrt{2\pi}} \int_c^{\infty} e^{-\frac{1}{2}(x - p\sigma\sqrt{T})^2} dx \\ &= S_0^p e^{p(r-\frac{\sigma^2}{2})T + \frac{1}{2}\sigma^2 p^2 T} \frac{1}{\sqrt{2\pi}} \int_{-d_1}^{\infty} e^{-\frac{y^2}{2}} dy \\ &= S_0^p e^{p(r-\frac{\sigma^2}{2})T + \frac{1}{2}\sigma^2 p^2 T} P(Z > -d_1) \\ &= S_0^p e^{p(r-\frac{\sigma^2}{2})T + \frac{1}{2}\sigma^2 p^2 T} N(d_1). \end{aligned}$$

This completes the proof. □

Theorem 2.4. *The European BSM formula for the standard power payoff function* $\max\{S_T^p - K, 0\}$ *is*

$$C(S_0, 0) = S_0^p e^{(p-1)(r + p\frac{\sigma^2}{2})T} N(d_1) - Ke^{-rT} N(d_2),$$

where $d_2 = d_1 - p\sigma\sqrt{T}$.

Proof. Let $C(S_0, 0)$ be the value of European call option at time 0. Then the present value of the expected payoff $\max\{S_T^p - K, 0\}$ will be $e^{-rT}E[\max\{S_T^p - K, 0\}]$. This should be exactly equal to $C(S_0, 0)$. Hence

$$\begin{aligned} C(S_0, 0) &= e^{-rT} E[\max\{S_T^p - K, 0\}] \\ &= e^{-rT} E[I(S_T^p - K)] \\ &= e^{-rT} E[IS_T^p] - Ke^{-rT} E[I] \\ &= S_0^p e^{(p-1)(r + p\frac{\sigma^2}{2})T} N(d_1) - Ke^{-rT} N(d_2) \end{aligned}$$

This proves the result. □

Corollary 2.5. [5] *The European BSM formula for the plain vanilla payoff function* $\max\{S_T - K, 0\}$ *is*

$$C(S_0, 0) = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

where $d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$ *and* $d_2 = d_1 - \sigma\sqrt{T}$.

Proof. Take $p = 1$ in Theorem 2.4. □

3. BSM Formula for Exponential Option

In this section, we shall find BSM option pricing formulas for the exponential payoff function $\max\{e^{S_T} - e^K, 0\}$. Under the risk-neutral probabilities, e^{S_T} for the exponential option can be expressed as

$$e^{S_T} = e^{S_0 + rS_0T + \sigma S_0 \sqrt{T} Z}, \tag{3.1}$$

where Z is a standard normal random variable. Let I be the indicator random variable for the exponential option. Then

$$I = \begin{cases} 1, & \text{if } S_T > K \\ 0, & \text{if } S_T \leq K. \end{cases} \tag{3.2}$$

Lemma 3.1. *Let* Z *and* I *be as in Equations* (3.1)*and* (3.2)*, respectively. Then*
(1) The indicator random variable can be expressed as

$$I = \begin{cases} 1, & \text{if } Z > \sigma S_0\sqrt{T} - d_1 \\ 0, & \text{otherwise}, \end{cases}$$

where $d_1 = \frac{S_0 - K + (\sigma^2 S_0 + r)S_0 T}{\sigma S_0 \sqrt{T}}$.
(2) The expected value $E[I] = N(d_1 - \sigma S_0\sqrt{T})$.

Proof. Note that

$$\begin{aligned} e^{S_T} > e^K &\Leftrightarrow e^{S_0 + rS_0T + \sigma S_0\sqrt{T}Z} > e^K \\ &\Leftrightarrow S_0 + rS_0T + \sigma S_0\sqrt{T}Z > K \\ &\Leftrightarrow Z > \frac{K - S_0 - rS_0T}{\sigma\sqrt{T}} \\ &\Leftrightarrow Z > \sigma S_0\sqrt{T} - d_1. \end{aligned}$$

This proves (1). Now

$$\begin{aligned} E[I] &= P(e^{S_T} > e^K) \\ &= P(Z > \sigma S_0\sqrt{T} - d_1) \\ &= P(Z < d_1 - \sigma S_0\sqrt{T}) \\ &= N(d_1 - \sigma S_0\sqrt{T}) \end{aligned}$$

This proves (2). □

Theorem 3.2. *The European BSM formula for the exponential payoff function*
$\max\{e^{S_T} - e^K, 0\}$ *is*

$$C(S_0, 0) = e^{-rT + S_0 + rS_0T + \frac{1}{2}\sigma^2 S_0^2 T} N(d_1) - e^K e^{-rT} N(d_2),$$

where $d_2 = d_1 - \sigma S_0\sqrt{T}$.

Proof. First we show that

$$E[Ie^{S_T}] = e^{S_0+rS_0T+\frac{1}{2}\sigma^2S_0^2T}N(d_1).$$

In order to this, take $c = \sigma S_0\sqrt{T} - d_1$. It follows from Lemma 3.1 that

$$\begin{aligned} E[Ie^{S_T}] &= \int_c^\infty e^{S_0+rS_0T+\sigma S_0\sqrt{T}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\ &= \frac{e^{S_0+rS_0T}}{\sqrt{2\pi}}\int_c^\infty e^{-\frac{1}{2}(x^2-2\sigma S_0\sqrt{T}x)}dx \\ &= \frac{e^{S_0+rS_0T+\frac{1}{2}\sigma^2S_0^2T}}{\sqrt{2\pi}}\int_c^\infty e^{-\frac{1}{2}(x-\sigma S_0\sqrt{T})^2}dx \\ &= e^{S_0+rS_0T+\frac{1}{2}\sigma^2S_0^2T}\frac{1}{\sqrt{2\pi}}\int_{-d_1}^\infty e^{-\frac{y^2}{2}}dy \\ &= e^{S_0+rS_0T+\frac{1}{2}\sigma^2S_0^2T}P(Z > -d_1) \\ &= e^{S_0+rS_0T+\frac{1}{2}\sigma^2S_0^2T}N(d_1). \end{aligned}$$

Let $C(S_0, 0)$ be the value of European call option at time 0. Then the present value of the expected payoff $\max\{e^{S_T} - e^K, 0\}$ will be $e^{-rT}E[\max\{e^{S_T} - e^K, 0\}]$. This should be exactly equal to $C(S_0, 0)$. Hence

$$\begin{aligned} C(S_0, 0) &= e^{-rT}E[\max\{e^{S_T} - e^K, 0\}] \\ &= e^{-rT}E[I(e^{S_T} - e^K)] \\ &= e^{-rT}E[Ie^{S_T}] - e^{K-rT}E[I] \\ &= e^{-rT+S_0+rS_0T+\frac{1}{2}\sigma^2S_0^2T}N(d_1) - e^{K-rT}N(d_2) \end{aligned}$$

This completes the proof. □

Remarks 3.3.

(I) By similar arguments, we can derive BSM formulas for European put option for standard power payoff function $\max\{K - S_T^p, 0\}$ and exponential payoff function $\max\{e^K - e^{S_T}, 0\}$ using probabilistic approach.

(II) In principle, we should be able to derive BSM formulas using the probabilistic approach for those payoff functions for which the BSM formulas are derived using BSM differential equation. However, it seems difficult to derive BSM formulas for some payoff functions. For example, we find difficulties for the log payoff function $\max\{\ln(\frac{S_T}{K}), 0\}$ and the modified log payoff function $\max\{S_T\ln(\frac{S_T}{K}), 0\}$.

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