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Numerical Solution of BSM Equation Using Some Payoff Functions

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Abstract

In this paper we have derived an iterative finite difference formula to get numerical solution of Black-Scholes-Merton (BSM) Partial Differential Equation using Plain Vanila Payoff and Log Payoff functions. The solution of BSM equation represents the model for pricing an option (i.e Call/Put). It should be noted that most of the trading platforms use BSM equation along with The Plain Vanila Payoff function.

Keywords: Black-Scholes-Merton model; Heat Equation; Finite Difference Formulas; Taylor's Series Expansion

AMS Subject Classification (2010): 35K05; 58J35; 65M06

1 Introduction

The present day numerical computations were almost unknown before 1950. The high speed computing machine (the computer) has made possible the solution of problems having great complexity in mathematics as well as in financial mathematics [1]. Here in this paper we have used MATHEMATICA to get numerical solutions of BSM euqation. Also we compare these solutions with MATHEMATICA exact value of the analytical solution of a heat equation which is obtained by applying some transformations to BSM equation.

For the usual theory (i.e. existence, uniqueness, differentiability of solution etc.) related to the partial differential equations we refer to [4] and [2].

In Financial Mathematics, the Black-Scholes-Merton equation is used to find the value of European Call/Put options.

The BSM equation [5] is,

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{1.1}$$

Here $V(S,t)$ is the value of European Call option.

S= Spot price of asset (i.e. the price of asset at time $t = 0$)

r= Risk free interest rate

σ= Volatility

X= Exercise price or Strike price

T= Total period of time

Here we consider the European Call option whose final payoff at the expiry time T is given by a function $f(S)$.
i.e.

$$\lim_{t \to T^-} V(S,t) = f(S)$$

As it is assumed here that $V(S,t)$ is a continuous function,

$$\lim_{t \to T^-} V(S,t) = V(S,T)$$

Therefore the final payoff at the expiry time is given by,

$$V(S,T) = f(S)$$

Here, in section-2 we have applied some transformations to BSM equation to get a Heat equation as in [3] and the obtained analytical solution of this Heat equation will be used to compare with our numerical solutions in section-4.

In section-3 we have converted the BSM equation into an equation with constant coefficients by applying a transformation and the idea of numerical method for solution of BSM equation is given in detail which is based on some discussions in [1] and [7] . A new Finite Difference Formula is also derived in this section.

In section-4 numerical computations are carried out using the formula that we obtain in section-3 to get numerical solutions of BSM equation and have compared it with analytical solution of the Heat equation derived in section-2, which demonstrates the effect of our finite difference formula.

2 Heat eqaution form of BSM equation

The equation (1.1),

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = 0$$

is known as Black-Scholes-Merton Partial Differential Equation.
where,

$$V(S,T) = f(S)$$

We convert this equation into Heat Equation as follows [3]:
We let,

$$\begin{aligned} y &= T - t \\ x &= \ln\left[\frac{S}{X}\right] + \left(r - \frac{\sigma^2}{2}\right)(T - t) \end{aligned} \tag{2.1}$$

$$D(x,y) = e^{r(T-t)}V(S,t)$$

Hence the function $V(S,t)$ is replaced by the function $D(x,y)$ in equation (1.1) and the resulting equation is a Heat equation,

$$\frac{\partial D}{\partial y} = \frac{\sigma^2}{2}\frac{\partial^2 D}{\partial x^2}. \tag{2.2}$$

Also the boundary condition, $\lim_{t\to T^-} V(S,t) = f(S)$ is converted into the initial condition,

$$\lim_{y\to 0^+} D(x,y) = f(Xe^x)$$

The analytical solution of this initial value problem of Heat Equation is obtained using either Method of Separation of Variables [8] or Fourier Transforms [6].
The solution obtained is,

$$D(x,y) = \frac{1}{\sigma\sqrt{2\pi y}} \int_{-\infty}^{\infty} f(\tau) e^{\frac{-(x-\tau)^2}{2\sigma^2 y}} \mathrm{d}\tau \tag{2.3}$$

We will compare numerical solution obtained by our finite difference formula with this analytical solution (2.3).

3 Finite Difference Formula for BSM Equation

Here, first of all we will convert the BSM equation into an equation with constant coefficients as follows:
We let,

$$V(S,t) = u(w,t)$$

where, $w = \text{Log}\left[\frac{S}{X}\right]$. Therefore,

$$V(S,T) = f(S) \Rightarrow u(w,T) = f(Xe^w).$$

Hence we get,

$$V_t = u_t$$
$$V_S = u_w w_s = \tfrac{1}{S} u_w$$
$$V_{SS} = u_w w_{ss} + (w_s)^2 u_{ww} = -\tfrac{1}{S^2} u_w + \tfrac{1}{S^2} u_{ww}.$$

Now, applying these expressions to (1.1), we get,

$$u_t + \frac{\sigma^2}{2}(u_{ww} - u_w) + r u_w - r u = 0 \tag{3.1}$$

We apply finite difference formulas to above (3.1) as follows:
By Taylor series expansion,

$$u(w+h,t) = u(w,t) + h u_w(w,t) + \tfrac{1}{2!} h^2 u_{ww}(w,t) + \dots$$
$$\&$$
$$u(w-h,t) = u(w,t) - h u_w(w,t) + \tfrac{1}{2!} h^2 u_{ww}(w,t) + \dots$$

Adding and subtracting the above two, we get,

$$u(w+h,t) + u(w-h,t) = 2u(w,t) + h^2 u_{ww}(w,t) + O(h^4)$$

$$\&$$

$$u(w+h,t) - u(w-h,t) = 2h u_w(w,t) + O(h^3)$$

where $O(h^n)$ denotes the terms containing h^n and higher powers of h.
Hence,

$$u_w = \frac{u(w+h,t) - u(w-h,t)}{2h} + O(h^2)$$
$$\&$$
$$u_{ww} = \frac{u(w+h,t) - 2u(w,t) + u(w-h,t)}{h^2} + O(h^2).$$

As we want to use the scheme centred in space and forward in time, we will use the finite difference formulas,

$$u_w = \frac{u(w+h,t+k) - u(w-h,t+k)}{2h} + O(h^2)$$
$$\&$$
$$u_{ww} = \frac{u(w+h,t+k) - 2u(w,t+k) + u(w-h,t+k)}{h^2} + O(h^2).$$

Also by Taylor series expansion,
$u(w,t+k) = u(w,t) + ku_t(w,t) + \frac{k^2}{2!}u_{tt}(w,t) + \frac{k^3}{3!}u_{ttt}(w,t) + O(k^4)$
Hence,

$$u_t = \frac{u(w,t+k) - u(w,t)}{k} + O(k^2).$$

In the above expressions, taking $w = ih$ & $t = jk$ and moreover writing;

$$\begin{aligned} u(w,t) &= u(ih,jk) \\ &= u_{i,j} \\ u(w+h,t) &= u((i+1)h,jk) \\ &= u_{i+1,j} \\ u(w,t+k) &= u(ih,(j+1)k) \\ &= u_{i,j+1} \end{aligned}$$

we get,

$$\begin{aligned} u_t &= \frac{u_{i,j+1} - u_{i,j}}{k} + O(k^2) \\ u_w &= \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h} + O(h^2) \\ u_{ww} &= \frac{u_{i+1,j+1} - 2u_{i,j} + u_{i-1,j+1}}{h^2} + O(h^2). \end{aligned}$$

Using above expressions, (3.1) is rewritten as,

$$\frac{u_{i,j+1} - u_{i,j}}{k} + \frac{\sigma^2}{2}\left[\frac{u_{i+1,j+1} - 2u_{i,j} + u_{i-1,j+1}}{h^2} - \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h}\right] + r\left[\frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h}\right] - ru_{i,j} + O(h^2 + k^2) = 0$$

Therefore,

$$\left[\frac{\sigma^2}{2h^2} - \frac{\sigma^2}{4h} + \frac{r}{2h}\right]u_{i+1,j+1} + \left[\frac{1}{k} - \frac{\sigma^2}{h^2}\right]u_{i,j+1} + \left[\frac{\sigma^2}{2h^2} + \frac{\sigma^2}{4h} - \frac{r}{2h}\right]u_{i-1,j+1} + O(h^2 + k^2) = \left[\frac{1}{k} + r\right]u_{i,j}.$$

Multiplying both the sides of above expression by k, we get,

$$\left[\frac{\sigma^2}{2}\left(\frac{k}{h^2}\right) - \frac{\sigma^2}{4}\left(\frac{k}{h}\right) + \frac{r}{2}\left(\frac{k}{h}\right)\right]u_{i+1,j+1} + \left[1 - \sigma^2\left(\frac{k}{h^2}\right)\right]u_{i,j+1} + \left[\frac{\sigma^2}{2}\left(\frac{k}{h^2}\right) + \frac{\sigma^2}{4}\left(\frac{k}{h}\right) - \frac{r}{2}\left(\frac{k}{h}\right)\right]u_{i-1,j+1} + O(h^2k + k^3) = (1 + rk)\,u_{i,j}. \quad (3.2)$$

In above (3.2), if we take $\frac{k}{h} = \mu$ and $\frac{k}{h^2} = \xi$, it is rewritten as,

$$\left[\frac{\sigma^2}{2}\xi - \frac{\sigma^2}{4}\mu + \frac{r}{2}\mu\right]u_{i+1,j+1} + \left[1 - \sigma^2\xi\right]u_{i,j+1} + \left[\frac{\sigma^2}{2}\xi + \frac{\sigma^2}{4}\mu - \frac{r}{2}\mu\right]u_{i-1,j+1} + O(h^2k + k^3) = (1 + rk)\,u_{i,j}. \quad (3.3)$$

Above (3.3) is the required Finite Difference Formula for BSM equation which works explicitly for given condition $u[w,T] = f(Xe^w)$.
Note that, this same formula may work implicitly if the initial condition is given as $u[w,0] = f(w)$.

4 Solution of BSM equation using some payoff functions

The BSM equation is,

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = 0$$

with the condition, $\lim_{t\to T^-} V(S,t) = f(S)$.

1) Plain Vanila Payoff Function

For Plain Vanila Payoff,

$$\begin{aligned} f(S) &= S - X,\ S \geq X \\ &= 0 \qquad\quad ,S \leq X \end{aligned}$$

Here we only consider the case when $S \geq X$.
Therefore, as $S = Xe^w$

$$\begin{aligned} & Xe^w \geq X \\ \Rightarrow \quad & w \geq \text{Log}[1] \\ \Rightarrow \quad & w \geq 0 \end{aligned}$$

Also,

$$\begin{aligned} f(S) &= S - X \\ \Rightarrow u[w,T] &= Xe^w - X \\ \Rightarrow u[w,T] &= X(e^w - 1) \end{aligned}$$

Here we will consider, $X = 40,\ \sigma = 0.2,\ T = 0.5,\ \&\ r = 0.1$, where, X is strike price, σ is volatility, T is expiry time and r is rate of interest.
For different values of S we will compute the values of x & w and then compare the numerical solution $u[w,t]$ obtained by implementing (3.3) and analytical solution $D[x,y]$ (i.e.(2.3)) in Mathematica.
Note that, we are considering $T = 0.5$ and computing solutions for the last to second time row (i.e. $t = 0.499$, $\because\ k = 0.001$ and $T - k = 0.499$) and therefore we get,

$$y = T - t = 0.001$$

Table 1: S=41

$w\ (= \text{Log}\left[\frac{S}{X}\right])$	$\text{Log}\left[\frac{41}{40}\right]$
x (*see* (2.1))	0.0247726
Numerical Solution using (3.3) implemented on Mathematica	$u[\text{Log}\left[\frac{41}{40}\right], 0.499] = 1.0040$
Mathematica Exact Value of The Analytical Solution (2.3)	$D[0.0247726, 0.001] = \frac{X}{\sigma\sqrt{2\pi(0.001)}}\int_{-\infty}^{\infty}(e^{\tau}-1)e^{-\frac{(0.0247726-\tau)^2}{2\sigma^2(0.001)}}d\tau$
Difference	$0.970534 * 10^{-3}$

Table 2: S=42

w (= Log $\left[\frac{S}{X}\right]$)	Log $\left[\frac{42}{40}\right]$
x (*see* (2.1))	0.0488702
Numerical Solution using (3.3) implemented on Mathematica	$u[\text{Log}\left[\frac{42}{40}\right], 0.499] = 2.0040$
Mathematica Exact Value of The Analytical Solution (2.3)	$D[0.0488702, 0.001] = \frac{X}{\sigma\sqrt{2\pi(0.001)}}\int_{-\infty}^{\infty}(e^{\tau}-1)e^{-\frac{(0.0488702-\tau)^2}{2\sigma^2(0.001)}}d\tau$
Difference	$0.194308 * 10^{-3}$

2) Log Payoff Function

For Log Payoff,

$$f(S) = \text{Log}\left[\frac{S}{X}\right],\ S \geq X.$$

We use the same values of $\sigma,\ X,\ T,\ \&\ r$ as we have used earlier.

Also, as $S = Xe^w$

$$Xe^w \geq X$$
$$\Rightarrow \quad w \geq \text{Log}[1]$$
$$\Rightarrow \quad w \geq 0$$

And,

$$f(S) = \text{Log}\left[\frac{S}{X}\right]$$
$$\Rightarrow u[w, T] = \text{Log}\left[\frac{Xe^w}{X}\right]$$
$$\Rightarrow u[w, T] = \text{Log}\left[e^w\right]$$
$$\Rightarrow u[w, T] = w$$

Now for different values of S the comparison of Numerical solution $u[w, t]$ obtained by implementing (3.3) in Mathematica and analytical solution $D[x, y]$ (i.e. (2.3)) is given below.

Here also we are computing solutions for the last to second time row (i.e. $t = 0.499$, $\because\ k = 0.001$ and $T - k = 0.499$) and therefore we get,

$$y = T - t = 0.001$$

Table 3: S=41

w (= Log $\left[\frac{S}{X}\right]$)	Log $\left[\frac{41}{40}\right]$
x (*see* (2.1))	0.0247726
Numerical Solution using (3.3) implemented on Mathematica	$u[\text{Log}\left[\frac{41}{40}\right], 0.499] = 0.247701$
Mathematica Exact Value of The Analytical Solution (2.3)	$D[0.0247726, 0.001] = \frac{1}{\sigma\sqrt{2\pi(0.001)}}\int_{-\infty}^{\infty}\tau e^{-\frac{(0.0247726-\tau)^2}{2\sigma^2(0.001)}}d\tau$
Difference	$2.54212 * 10^{-6}$

Table 4: S=42

w (= Log $\left[\frac{S}{X}\right]$)	Log $\left[\frac{42}{40}\right]$
x (*see* (2.1))	0.0488702
Numerical Solution using (3.3) implemented on Mathematica	$u[\text{Log}\left[\frac{42}{40}\right], 0.499] = 0.0488653$
Mathematica Exact Value of The Analytical Solution (2.3)	$D[0.0488702, 0.001] = \frac{1}{\sigma\sqrt{2\pi(0.001)}}\int_{-\infty}^{\infty} f\tau e^{-\frac{(0.0488702-\tau)^2}{2\sigma^2(0.001)}}d\tau$
Difference	$4.88653 * 10^{-6}$

Note:

1. The parameters X, σ & r are considered on basis of their occurrence in trading terminals.
2. As the boundary condition $V(S,T) = f(S)$ depends on T, change in T produces change in the solution.
   For example,
   if we take $T = 0.6$, for the last to second time row
   $$y = T - t = T - (T - k) = k = 0.001,$$
   which is the same as we have obtained for $T = 0.5$.
   But when $T = 0.5$, the last to second time row is,
   $$t = T - k = 0.5 - 0.001 = 0.499$$
   and when $T = 0.6$, the last to second time row is,
   $$t = T - k = 0.6 - 0.001 = 0.599.$$
   Hence,
   $$T = 0.5 \quad \Rightarrow \quad V(S, 0.499) \approx D(x, 0.001),$$
   and
   $$T = 0.6, \quad \Rightarrow \quad V(S, 0.599) \approx D(x, 0.001).$$

Conclusion

We have derived (in section-3) a finite difference formula for Black-Scholes-Merton equation and (in section-4) have computed numerical solutions for Plain Vanila Payoff function and Log Payoff Function. Also compared the numerical solutions obtained by our finite difference formula with the analytical solution of heat equation which is obtained by applying some transformations to Black-Scholes-Merton Equation. The above tables indicate that the numerical solutions obtained by using the new Finite Difference Formula are very close to the analytical solutions.

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