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APPROXIMATE REGULARITY AND ITS AVATARS IN LAU PRODUCT OF COMMUTATIVE BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras, and let θ be a multiplicative linear functional on $\mathcal{B}$. Then θ induces a multiplication on the Cartesian space $\mathcal{A}\times\mathcal{B}$ making it a Banach algebra, which is denoted by $\mathcal{A}\times_\theta\mathcal{B}$. We shall examine the stability of weak regularity, approximate regularity and approximate normality of this Banach algebra.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let θ be a multiplicative linear functional on $\mathcal{B}$. Then the product space $\mathcal{A}\times\mathcal{B}$ is a Banach algebra with the product

$$(a,b)(c,d)=(ac+\theta(d)a+\theta(b)c,bd)\quad((a,b),(c,d)\in\mathcal{A}\times\mathcal{B})$$

and the norm

$$\|(a,b)\|_1=\|a\|_{\mathcal{A}}+\|b\|_{\mathcal{B}}\quad((a,b)\in\mathcal{A}\times\mathcal{B}).$$

This Banach algebra is called the Lau product of $\mathcal{A}$ and $\mathcal{B}$ and is denoted by $\mathcal{A}\times_\theta\mathcal{B}$. This product was introduced by Lau [7] for certain class of Banach algebras and was extended by Sangani Monfared [9] for the general case. Many Banach algebra properties of $\mathcal{A}\times_\theta\mathcal{B}$ are studied in [6, 11, 10]. The product $\mathcal{A}\times_\theta\mathcal{B}$ generalizes the unitification process of adjoining identity by taking $\mathcal{B}=\mathbb{C}$ and $\theta(\lambda)=\lambda$. Also, notice that $\mathcal{A}\times_\theta\mathcal{B}$ is commutative if and only if both $\mathcal{A}$ and $\mathcal{B}$ are commutative. In the present paper, we investigate the spectral properties of the commutative Banach algebra $\mathcal{A}\times_\theta\mathcal{B}$. These properties include Tauberian, weak regularity, approximate regularity and approximate normality.

2. Spectral Properties of $\mathcal{A}\times_\theta\mathcal{B}$

If $\mathcal{A}$ and $\mathcal{B}$ are commutative Banach algebras, then the Gelfand space $\Delta(\mathcal{A}\times_\theta\mathcal{B})$ of $\mathcal{A}\times_\theta\mathcal{B}$ is the union of the sets $E=\{(\varphi,\theta):\varphi\in\Delta(\mathcal{A})\}$ and $F=\{(0,\psi):\psi\in\Delta(\mathcal{B})\}$ [9]. Topologies on E and F are the subspace topologies. Let $\mathcal{A}$ be a commutative Banach algebra, and let $\mathcal{A}_e=\mathcal{A}\times\mathbb{C}$ be the unitization of $\mathcal{A}$. Then $\Delta(\mathcal{A}_e)=\{\widetilde{\varphi}:\varphi\in\Delta(\mathcal{A})\}\cup\{\varphi_\infty\}$, where $\widetilde{\varphi}(a,\lambda)=\varphi(a)+\lambda$ and $\varphi_\infty(a,\lambda)=\lambda$ for all $(a,\lambda)\in\mathcal{A}_e$.

Define $f:\Delta(\mathcal{A})\to E$ by $f(\varphi)=(\varphi,\theta)$, $g:\Delta(\mathcal{B})\to F$ by $g(\psi)=(0,\psi)$ and $\widetilde{f}:\Delta(\mathcal{A}_e)\to E\cup\{(0,\theta)\}$ by $\widetilde{f}(\widetilde{\varphi})=(\varphi,\theta)$ and $\widetilde{f}(\varphi_\infty)=(0,\theta)$.

Proposition 2.1. [4, Proposition 2.1.] *If $\mathcal{A}$, $\mathcal{B}$ are commutative Banach algebras and $\theta\in\Delta(\mathcal{B})$, then the functions f, $\widetilde{f}$ and g are homeomorphisms.*

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This result implies that $E \cup \{(0,\theta)\}$ is homeomorphic to $\Delta(\mathcal{A}_e)$ and F is homeomorphic to $\Delta(\mathcal{B})$.

If $\mathcal{A}$ is a commutative Banach algebra and $a \in \mathcal{A}$, then the *support* of $\widehat{a}$, *supp* $\widehat{a}$, is the the closure of the set $\{\varphi \in \Delta(\mathcal{A}) : \widehat{a}(\varphi) \neq 0\}$. The element $\widehat{a}$ has *compact support* if the set *supp* $\widehat{a}$ is a compact set.

Proposition 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras, $\theta \in \Delta(\mathcal{B})$, and let $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$. Then $\widehat{(a,b)}$ has compact support if and only if $\widehat{b}$ has so.*

Proof. We note that

$$\mathrm{supp}(\widehat{(a,b)}) = \overline{\{(\varphi,\theta) \in \Delta(\mathcal{A} \times_\theta \mathcal{B}) : \varphi(a) + \theta(b) \neq 0\}} \cup \overline{\{(0,\psi) \in \Delta(\mathcal{A} \times_\theta \mathcal{B}) : \psi(b) \neq 0\}}.$$

If $\widehat{(a,b)}$ has a compact support, then $\mathrm{supp}(\widehat{b}) = g^{-1}(\mathrm{supp}\widehat{(a,b)} \cap F)$ is compact, i.e., $\widehat{b}$ has compact support.

Conversely, assume that $\mathrm{supp}(\widehat{b})$ is compact. Since the set $f(\Delta(\mathcal{A})) \cup \{(0,\theta)\}$ is compact, $\overline{\{(\varphi,\theta) \in \Delta(\mathcal{A} \times_\theta \mathcal{B}) : \varphi(a) + \theta(b) \neq 0\}}$ is compact. Clearly, $g(\mathrm{supp}(\widehat{b})) = \overline{\{(0,\psi) \in \Delta(\mathcal{A} \times_\theta \mathcal{B}) : \psi(b) \neq 0\}}$ is compact. Therefore $\mathrm{supp}(\widehat{(a,b)})$ is compact. □

Definition 2.3. [5, Defn 5.1.7] *A commutative Banach algebra $\mathcal{A}$ is* Tauberian *if the set of all $a \in \mathcal{A}$ such that $\widehat{a}$ has a compact support is dense in $\mathcal{A}$.*

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras, and let $\theta \in \Delta(\mathcal{B})$. Then $\mathcal{A} \times_\theta \mathcal{B}$ is Tauberian if and only if $\mathcal{B}$ is Tauberian.*

Proof. Let $\mathcal{A} \times_\theta \mathcal{B}$ be Tauberian. Let $b_0 \in \mathcal{B}$ and $\epsilon > 0$. Then there is $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$ such that $\|(a,b) - (0,b_0)\|_1 = \|a\|_{\mathcal{A}} + \|b - b_0\|_{\mathcal{B}} < \epsilon$ and $\mathrm{supp}(\widehat{(a,b)})$ is compact. Then $\|b - b_0\|_{\mathcal{B}} < \epsilon$ and $\mathrm{supp}(\widehat{b})$ is compact by Proposition 2.2. Hence $\mathcal{B}$ is Tauberian.

Conversely, assume that $\mathcal{B}$ is Tauberian. Let $(a_0, b_0) \in \mathcal{A} \times_\theta \mathcal{B}$ and $\epsilon > 0$. Then there exists $b \in \mathcal{B}$ such that $\|b - b_0\|_{\mathcal{B}} < \epsilon$ and $\mathrm{supp}(\widehat{b})$ is compact. Then $\mathrm{supp}(\widehat{(a_0,b)})$ is compact and $\|(a_0,b_0) - (a_0,b)\|_1 = \|b_0 - b\|_{\mathcal{B}} < \epsilon$. Therefore $\mathcal{A} \times_\theta \mathcal{B}$ is Tauberian. □

Corollary 2.5. *If $\mathcal{A}$ is a commutative Banach algebra, then $\mathcal{A}_e$ is Tauberian.*

Definition 2.6. [2, 1] *A semisimple commutative Banach algebra $\mathcal{A}$ is* weakly regular *if given a proper closed subset U of $\Delta(\mathcal{A})$, there exists a nonzero element a of $\mathcal{A}$ with $\widehat{a}(\varphi) = 0$ for every $\varphi \in U$.*

Let X be a locally compact Hausdorff space, and let $\mathcal{A}$ be a subalgebra of $C_0(X)$ which strongly separates the points of X. A subset M of X is a *boundary* of $\mathcal{A}$ if for every $f \in \mathcal{A}$ there is $y \in M$ such that $|f(y)| = \sup\{|f(x)| : x \in X\} = \|f\|_\infty$. The intersection of all closed boundaries of $\mathcal{A}$, which is a boundary [5, Theorem 3.3.2], is called the Shilov boundary of $\mathcal{A}$. It is denoted by $\partial(\mathcal{A})$.
Let $\mathcal{A}$ be a commutative Banach algebra, and let $\Gamma : \mathcal{A} \to C_0(\Delta(\mathcal{A}))$, $\Gamma(a) = \widehat{a}$, be the Gelfand representation of $\mathcal{A}$. The set $\partial(\Gamma(\mathcal{A}))$ is called the *Shilov boundary* of $\mathcal{A}$ and is denoted by $\partial(\mathcal{A})$.

A norm $|\cdot|$ on a commutative Banach algebra $\mathcal{A}$ is a *uniform norm* if $|a^2| = |a|^2$ for all $a \in \mathcal{A}$. A Banach algebra has *unique uniform norm property* (UUNP) if it has exactly one uniform norm. A commutative Banach algebra $\mathcal{A}$ has *spectral extension property* if $r(a) \leq |a|$ $(a \in \mathcal{A})$ for every norm $|\cdot|$ on $\mathcal{A}$ [8]. Clearly, SEP implies UUNP. It is not known whether UUNP implies SEP or not [2].

We use the following results for showing the stability of weak regularity with respect to the Lau product.

Theorem 2.7 ([5], Corollary 4.6.7)**.** *For a semisimple commutative Banach algebra $\mathcal{A}$, $\mathcal{A}$ is weakly regular if and only if $\mathcal{A}$ has UUNP and satisfies $\partial(\mathcal{A}) = \Delta(\mathcal{A})$.*

Theorem 2.8. [4, Theorem 2.2] *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras, and let $\theta \in \Delta(\mathcal{B})$. Then $\partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\} = (f(\partial(\mathcal{A})) \cup g(\partial(\mathcal{B})) \setminus \{(0,\theta)\}$.*

Proposition 2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras, let $\theta \in \Delta(\mathcal{B})$, and let $(0, \theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$. Then $\varphi_\infty \in \partial(\mathcal{A}_e)$ or $(0, \theta) \in \overline{\partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0, \theta)\}}$.*

Lemma 2.10. [4, Lemma 3.4] *Let $\mathcal{A}$ be a semisimple commutative Banach algebra such that $\mathcal{A}$ has UUNP and $\partial(\mathcal{A})$ is compact subset of $\Delta(\mathcal{A})$. Then $\varphi_\infty \in \partial(\mathcal{A}_e)$ if and only if $\mathcal{A}$ has identity.*

Theorem 2.11. [4, Theorem 3.1.] *Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple commutative Banach algebras, and let $\theta \in \Delta(\mathcal{B})$.*

(i) *If $\mathcal{A}$ and $\mathcal{B}$ have UUNP, then $\mathcal{A} \times_\theta \mathcal{B}$ has UUNP.*

(ii) *Suppose that $\mathcal{A} \times_\theta \mathcal{B}$ has UUNP. Then $\mathcal{A}$ has UUNP. Moreover, if θ is continuous with respect to every uniform norm on $\mathcal{B}$, then $\mathcal{B}$ also has UUNP.*

Theorem 2.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras, and let $\theta \in \Delta(\mathcal{A})$.*

(i) *Suppose that $\mathcal{A} \times_\theta \mathcal{B}$ is weakly regular. Then $\mathcal{A}$ is weakly regular. Moreover if $\theta \in \partial(\mathcal{B})$ and it is continuous with respect to every norm on $\mathcal{B}$, then $\mathcal{B}$ is weakly regular.*

(ii) *If $\mathcal{A}$ and $\mathcal{B}$ are weakly regular, then $\mathcal{A} \times_\theta \mathcal{B}$ is weakly regular.*

Proof. (i) Since $\mathcal{A} \times_\theta \mathcal{B}$ is weakly regular, $\mathcal{A} \times_\theta \mathcal{B}$ has UUNP and $\partial(\mathcal{A} \times \mathcal{B}) = \Delta(\mathcal{A} \times_\theta \mathcal{B})$. By Theorem 2.11, $\mathcal{A}$ has UUNP and by Theorem 2.8, $\partial(\mathcal{A}) = \Delta(\mathcal{A})$. Hence $\mathcal{A}$ is weakly regular.

Since $\mathcal{A} \times_\theta \mathcal{B}$ is weakly regular, $\mathcal{A} \times_\theta \mathcal{B}$ has UUNP and $\partial(\mathcal{A} \times_\theta \mathcal{B}) = \Delta(\mathcal{A} \times_\theta \mathcal{B})$. By Theorem 2.8, $\partial(\mathcal{B}) \setminus \{\theta\} = \Delta(\mathcal{B}) \setminus \{\theta\}$. Since θ is continuous with respect to every norm on $\mathcal{B}$, $\mathcal{B}$ has UUNP. Since $\theta \in \partial(\mathcal{B})$, $\partial(\mathcal{B}) = \Delta(\mathcal{B})$. Hence $\mathcal{B}$ is weakly regular.

(ii) Since $\mathcal{A}$ and $\mathcal{B}$ are weakly regular, $\mathcal{A}$ and $\mathcal{B}$ has UUNP, $\partial(\mathcal{A}) = \Delta(\mathcal{A})$ and $\partial(\mathcal{B}) = \Delta(\mathcal{B})$. Hence $\mathcal{A} \times_\theta \mathcal{B}$ has UUNP and $\partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0, \theta)\} = \Delta(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0, \theta)\}$. If $(0, \theta) \in \overline{\partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0, \theta)\}}$, then $(0, \theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$. If $(0, \theta) \notin \overline{\partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0, \theta)\}}$, then by Proposition 2.9, $\phi_\infty \in \partial(\mathcal{A}_e)$ and by Theorem 2.10, $\mathcal{A}$ has identity. By [3, Lemma 2.6.], $(0, \theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$. This shows that $\mathcal{A} \times_\theta \mathcal{B}$ is weakly regular. □

Definition 2.13.

(i) *A commutative Banach algebra $\mathcal{A}$ is* regular *if given a closed subsets U of $\Delta(\mathcal{A})$, $\varphi \in \Delta(\mathcal{A}) \setminus U$, there exists $a \in \mathcal{A}$ such that $\varphi(a) = 1$ and $\widehat{a}(U) = \{0\}$.*

(ii) *A commutative Banach algebra $\mathcal{A}$ is* approximately regular *if given a closed set $U \subset \Delta(\mathcal{A})$, $\varphi \in \Delta(\mathcal{A}) \setminus U$ and $\epsilon > 0$ there exists $a \in \mathcal{A}$ such that $\varphi(a) = 1$ and $\sup\{|\widehat{a}(\psi)| : \psi \in U\} < \epsilon$.*

Theorem 2.14. *Let $\mathcal{A}$ be a unital, semisimple, commutative Banach algebra. If $\mathcal{A}$ is approximately regular, then every closed ideal I of $\mathcal{A}$ is approximately regular.*

Proof. Let $\varphi \in \Delta(I)$, U be any open set of $\Delta(I)$ containing it, and let $\epsilon > 0$. Then there exists $i \in I$ such that $\varphi(i) = 1$. Since $\Delta(I) = \Delta(\mathcal{A}) \setminus h(I)$ [5, Lemma 2.2.15], U is an open subset of $\Delta(\mathcal{A})$. As $\mathcal{A}$ is approximately regular, there is $a \in \mathcal{A}$ such that $\varphi(a) = 1$ and $\|\widehat{a}|_{U^c}\|_\infty < \frac{\epsilon}{\|\widehat{i}|_{U^c}\|_\infty}$. Take $b = ai$. Then $\varphi(b) = \varphi(ai) = \varphi(a)\varphi(i) = 1$ and $\|\widehat{b}|_{U^c}\|_\infty = \|\widehat{ai}|_{U^c}\|_\infty \leq \|\widehat{a}|_{U^c}\|_\infty \|\widehat{i}|_{U^c}\|_\infty < \epsilon$. □

Theorem 2.15. *Let $\mathcal{A}$ be approximately regular commutative Banach algebra and I be a closed ideal in $\mathcal{A}$. Then I and $\mathcal{A}/I$ are approximately regular.*

Proof. We have $\Delta(I) = \Delta(\mathcal{A}) \setminus \mathrm{hull}(I)$ and $\Delta(\mathcal{A}/I) = \mathrm{hull}(I)$ with subspace topology. Let $\mathcal{A}$ be approximately regular. By Theorem 2.14, I is approximately regular. Let $\varphi \in \Delta(\mathcal{A}/I) = \mathrm{hull}(I)$, U be any open set of $\Delta(\mathcal{A}/I)$ containing it, and let $\epsilon > 0$. Since $\varphi \in \mathcal{A}$ and $U = V \cap \mathrm{hull}(I)$ for some open subset V of $\Delta(\mathcal{A})$, we have $a \in \mathcal{A}$ such that $\varphi(a) = 1$ and $\|\widehat{a}|_{V^c}\|_\infty < \epsilon$. Then for $a + I \in \mathcal{A}/I$ we have $\varphi(a + I) = 1$ and $\|\widehat{a + I}|_{U^c}\|_\infty < \epsilon$. Hence $\mathcal{A}/I$ is approximately regular. □

Theorem 2.16. *Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple commutative Banach algebras. Then $\mathcal{A}\times_\theta\mathcal{B}$ is approximately regular if and only if $\mathcal{A}_e$ and $\mathcal{B}$ are approximately regular.*

Proof. Let $\mathcal{A}\times_\theta\mathcal{B}$ be approximately regular. Let $\varphi_\infty \neq \widetilde{\varphi} \in \Delta(\mathcal{A}_e)$, U be an open subset of $\Delta(\mathcal{A}_e)$ containing it, and let $\epsilon > 0$. Let V be an open subset of $\Delta(\mathcal{A})$ containing φ and not containing φ_∞. Since $\widetilde{f}(U\cap V)$ is open subset of $\Delta(\mathcal{A}\times_\theta\mathcal{B})$, there exists $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$ such that $(\varphi,\theta)(a,b)=1$ and $\|\widehat{(a,b)}|_{\widetilde{f}(U\cap V)^c}\|_\infty \leq \epsilon$. The element $a+\theta(b)$ is in $\mathcal{A}_e$. We see that $\widetilde{\varphi}(a+\theta(b))=1$ and $\|\widehat{(a+\theta(b))}|_{U^c}\|_\infty \leq \epsilon$. Let U be an open subset of $\Delta(\mathcal{A}_e)$ containing φ_∞. Then there exists $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$ such that $(0,\theta)(a,b)=1$ and $\|\widehat{(a,b)}|_{(\widetilde{f}(U)\cup g(\Delta(\mathcal{B})))^c}\|_\infty \leq \epsilon$. The element $a+\theta(b)$ is in $\mathcal{A}_e$. Also, $\varphi_\infty(a+\theta(b))=1$ and $\|\widehat{(a,\theta(b))}|_{U^c}\|_\infty \leq \epsilon$. Therefore $\mathcal{A}_e$ is approximately regular.

Let $\psi\in\Delta(\mathcal{B})$, U be an open subset of $\Delta(\mathcal{B})$ containing it, and let $\epsilon > 0$. Then there exists $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$ such that $(0,\psi)(a,b)=1$ and $\|\widehat{(a,b)}|_{(g(U)\cup f(\Delta(\mathcal{A})))^c}\|_\infty \leq \epsilon$. Hence $\psi(b)=1$ and $\|\widehat{b}|_{U^c}\|_\infty \leq \epsilon$. Therefore $\mathcal{B}$ is approximately regular.

Conversely, assume that $\mathcal{A}_e$ and $\mathcal{B}$ are approximately regular. Let $(\varphi,\theta)\in\Delta(\mathcal{A}\times_\theta\mathcal{B})$, U be an open subset of $\Delta(\mathcal{A}\times_\theta\mathcal{B})$ containing it, and let $\epsilon>0$. Let V be an open neighborhood of (ϕ,θ) contained in E. Then there exists $a\in\mathcal{A}$ such that $\|\widehat{a}|_{f^{-1}(U\cap V)^c}\|_\infty < \epsilon$ and $\phi(a)=1$. Then $\widehat{(a,0)}(\varphi,\theta)=1$ and $\|\widehat{(a,0)}|_{U^c}\|_\infty \leq \|\widehat{(a,0)}|_{(U\cap V)^c}\|_\infty = \|\widehat{a}|_{f^{-1}(U\cap V)^c}\|_\infty < \epsilon$.

Let $(0,\theta)\neq(0,\psi)\in\Delta(\mathcal{A}\times_\theta\mathcal{B})$, U be an open subset of $\Delta(\mathcal{A}\times_\theta\mathcal{B})$ containing it, and let $\epsilon>0$. Consider an open subset $\widetilde{U}\subset U$ such that $\widetilde{U}\cap E=\emptyset$. Then there exists $b\in\mathcal{B}$ such that $\psi(b)=1$ and $\|\widehat{b}|_{g^{-1}(\widetilde{U})^c}\|_\infty \leq \epsilon$. Then $(0,b)\in\mathcal{A}\times_\theta\mathcal{B}$, $(0,\psi)=(0,b)=1$ and $\|\widehat{(0,b)}|_{U^c}\|_\infty \leq \|\widehat{(0,b)}|_{\widetilde{U}^c}\|_\infty \leq \epsilon$.

Let U be an open subset of $\Delta(\mathcal{A}\times_\theta\mathcal{B})$ containing $(0,\theta)$, and let $\epsilon>0$. Then there exists $(a,\lambda)\in\mathcal{A}_e$ such that $\varphi_\infty(a,\lambda)=1$ and $\|\widehat{(a,\lambda)}|_{\widetilde{f}(U)^c}\|_\infty \leq \epsilon$; and there exists $b\in\mathcal{B}$ such that $\theta(b)=\lambda$ and $\|\widehat{b}|_{g^{-1}(U)^c}\|_\infty \leq \epsilon$. Then $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$, $(0,\theta)(a,b)=1$ and $\|\widehat{(a,b)}|_{U^c}\|_\infty \leq \epsilon$. Hence $\mathcal{A}\times_\theta\mathcal{B}$ is approximately regular. ☐

Definition 2.17. *A commutative Banach algebra $\mathcal{A}$ is* normal *if given disjoint closed subsets U and V of $\Delta(\mathcal{A})$ there exists $a\in\mathcal{A}$ such that $\widehat{a}(U)=\{0\}$ and $\widehat{a}(V)=\{1\}$.*

By [5, Theorem 4.2.9], a commutative Banach algebra is normal if and only if it is regular.

The following approximate analogue of normality is analogous to the above, must have been considered in the literature. However, we have failed to identify the source.

Definition 2.18. *A commutative Banach algebra $\mathcal{A}$ is* approximately normal *if for any two disjoint closed sets U,V of $\Delta(\mathcal{A})$ and $\epsilon>0$ there exists $a\in\mathcal{A}$ such that* $\sup\{|\widehat{a}(\varphi)| : \varphi\in U\}<\epsilon$ *and* $\sup\{|\widehat{a}(\varphi)-1| : \varphi\in V\}<\epsilon$.

We shall give an example, Example 2.20, of a commutative Banach algebra which is regular (hence approximately regular) but not approximately normal.

Theorem 2.19. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras. If $\mathcal{A}$ and $\mathcal{B}$ are approximately normal, then $\mathcal{A}\times_\theta\mathcal{B}$ is approximately normal.*

Proof. For a closed subset U of $\Delta(\mathcal{A}\times_\theta\mathcal{B})$, let $U_\mathcal{A}=\{\varphi : (\varphi,\theta)\in U\}$, and $U_\mathcal{B}=\{\psi : (0,\psi)\in U\}$. Let $\epsilon>0$ and U,V be two disjoint closed subsets of $\Delta(\mathcal{A}\times_\theta\mathcal{B})$. Then $U_\mathcal{A}=f^{-1}(U)$ and $V_\mathcal{A}=f^{-1}(V)$. Since f is homeomorphism, $U_\mathcal{A}$ and $V_\mathcal{A}$ are disjoint closed subsets of $\Delta(\mathcal{A})$. Similarly, $U_\mathcal{B}$ and $V_\mathcal{B}$ are disjoint closed subsets of $\Delta(\mathcal{B})$. Then we have the following three cases.

(i) $(0,\theta)\notin U\cup V$.

Then there are elements $a\in\mathcal{A}$ and $b\in\mathcal{B}$ such that $\sup\{|\varphi(a)-1| : \varphi\in U_\mathcal{A}\}<\epsilon$, $\sup\{|\varphi(a)| : \varphi\in V_\mathcal{A}\}<\epsilon$, $\sup\{|\psi(b)-1| : \psi\in U_\mathcal{B}\}<\epsilon$ and $\sup\{|\psi(b)| : \psi\in V_\mathcal{B}\cup\{\theta\}\}<\epsilon$. Therefore $\sup\{|\varphi(a,b)-1| : \varphi\in U\}<2\epsilon$ and $\sup\{|\varphi(a,b)| : \varphi\in V\}<2\epsilon$.

(ii) $(0,\theta) \in U$.

Then there are elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $\sup\{|\varphi(a)| : \varphi \in U_{\mathcal{A}}\} < \epsilon$, $\sup\{|\varphi(a)-1| : \varphi \in V_{\mathcal{A}}\} < \epsilon$, $\sup\{|\varphi(b)-1| : \varphi \in U_{\mathcal{B}}\} < \epsilon$ and $\sup\{|\varphi(b)| : \varphi \in V_{\mathcal{B}}\} < \epsilon$. So, $\sup\{|\varphi(-a,b)-1| : \varphi \in U\} < 2\epsilon$ and $\sup\{|\varphi(-a,b)| : \varphi \in V\} < 2\epsilon$.

(iii) $(0,\theta) \in V$.

Then there are elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $\sup\{|\varphi(a)-1| : \varphi \in U_{\mathcal{A}}\} < \epsilon$, $\sup\{|\varphi(a)| : \varphi \in V_{\mathcal{A}}\} < \epsilon$, $\sup\{|\varphi(b)-1| : \varphi \in U_{\mathcal{B}}\} < \epsilon$ and $\sup\{|\varphi(b)| : \varphi \in V_{\mathcal{B}}\} < \epsilon$. These imply $\sup\{|\varphi(a,b)-1| : \varphi \in U\} < 2\epsilon$ and $\sup\{|\varphi(a,b)| : \varphi \in V\} < 2\epsilon$. Hence $\mathcal{A} \times_\theta \mathcal{B}$ is approximately normal. □

Example 2.20. *Converse of Theorem 2.19 is not true in general.*

Consider the convolution algebra $\mathcal{A} = L^1(\mathbb{T})$. Then $\mathcal{A}$ is regular. The Gelfand space $\Delta(\mathcal{A})$ is homeomorphic to $\mathbb{Z}$ via the map $n \mapsto \varphi_n$ from $\mathbb{Z}$ to $\Delta(\mathcal{A})$, where

$$\varphi_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}dx \qquad (f \in L^1(\mathbb{T})).$$

Note that the Gelfand topology on $\Delta(\mathcal{A})$ is discrete. Let $U = \{2n : n \in \mathbb{Z}\}$ and $V = \{2n+1 : n \in \mathbb{Z}\}$. Then both U and V are closed in $\Delta(\mathcal{A})$. It follows from Riemann-Lebesgue Lemma that there is no $f \in L^1(\mathbb{T})$ such that $|\widehat{f}(2n) - 1| \geq \frac{1}{2}$ for all $n \in \mathbb{Z}$.

Therefore $\mathcal{A} = L^1(\mathbb{T})$ is not approximately normal but it is regular.

Now, we show that the unitization $\mathcal{A}_e$ of $\mathcal{A}$ is not approximately normal. Note that $\Delta(\mathcal{A}_e)$ is one point compactification of $\Delta(A)$. Let U and V be two disjoint closed sets.

(i) *If $\varphi_\infty \in U$, then V is a finite set. Consider $f(x) = \sum_{\varphi_n \in V} e^{inx}$. Then $\widehat{(f,0)}|_V = 1$ and $\widehat{(f,0)}|_U = 0$.*

(ii) *If $\varphi_\infty \in V$, then U is a finite set. Take $f(x) = -\sum_{\varphi_n \in U} e^{inx}$. Then $\widehat{(f,1)}|_V = 1$ and $\widehat{(f,1)}|_U = 0$*

(iii) *If $\varphi_\infty \notin U$ and $\varphi_\infty \notin V$, then both U and V are finite sets. Take $f(x) = \sum_{\varphi_n \in V} e^{inx}$. Then $\widehat{(f,0)}|_V = 1$ and $\widehat{(f,0)}|_U = 0$*

This proves our claim.

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