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Some Graphs in the Context of Product Cordial Labeling

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Abstract

A graph $G = (V(G), E(G))$ is said to be product cordial graph if there exists a function $f : V(G) \to \{0,1\}$ with each edge uv assign the label $f(u)f(v)$, such that the number of vertices with label 0 and the number of vertices with label 1 differ atmost by 1, and the number of edges with label 0 and the number of edges with label 1 differ by atmost 1. We have discussed that a book graph $B_{m,n}$ is a product cordial graph if and only if m and n both are odd and $m \geq 3$. We have derived that book graph is a product cordial graph in context of duplication of some graph elements. We have proved that some edge deleted subgraphs of book graph is a product cordial graph. We have also derived that duplication of each of the vertices of a product cordial graph with an edge is a product cordial graph if the graph has even number of vertices. We have also proved that duplication of each of the vertices of a product cordial graph with an edge is product cordial graph if the graph has odd number of vertices and even number of edges.

Keywords: Product cordial labeling, book graph, triangular book graph, duplication, vertex switching.
AMS Subject Classification (2010): 05C78

1 Introduction

We begin with simple, finite, undirected graph $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are vertex set and edge set respectively. For all other terminology we follow Gross [2]. Now we provide brief summary of definitions and other information which are necessary for the present investigations.

Definition 2.1: If the vertices or edges or both of a graph are assigned values subject to certain conditions then it is known as *vertex or edge or total labeling* respectively.
For latest survey on graph labeling we refer to Gallian [1]. Vast amount of literature is available on different types of graph labeling and more than 2000 papers have been published.

Definition 2.2: A mapping $f : V(G) \to \{0,1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the label of the vertex v of G under f.
We denote $v_f(0)$ as the number of vertices with label 0 and $v_f(1)$ as the number of vertices with label 1.

Definition 2.3: Let $f : V(G) \to \{0,1\}$ for each edge uv assign the label $|f(u) - f(v)|$, then f is said to be a *cordial labeling* of G if the number of vertices with label 0 and the number of vertices with label 1 differ atmost by 1, and the number of edges with label 0 and the number of edges with label 1 differ by atmost 1.

Definition 2.4: A *product cordial labeling* of graph G with vertex set V is a function $f: V \to \{0,1\}$ such that each edge uv is assigned the label $f(u)f(v)$, the number of vertices with label 0 and the number of vertices with label 1 differ by atmost 1 and the number of edge with label 0 and the number of edge with label 1 differ by atmost 1. A graph which admits product cordial labeling is called a *product cordial*.

The notion of product cordial labeling was introduced by Sundaram, Ponraj and Somasundaram [4]. They proved that following graphs are product cordial: tree, unicyclic graph of odd order, triangular snakes, dragons and helms. Few results of product cordial graph in the context of some graph operations on gear graph are discussed by Prajapati and Raval [3].

Definition 2.5: The *neighborhood* of a vertex v of a graph is the set of all vertices adjacent to v. It is denoted by $N(v)$.

Definition 2.6: *Duplication of a vertex* of the graph G is the graph $G^{'}$ obtained from G by adding a new vertex $v^{'}$ to G such that $N(v^{'}) = N(v)$.

Definition 2.7: *Duplication of a vertex v_k by a new edge $e = v_k^{'}v_k^{''}$* in a graph G produces a new graph $G^{'}$ such that $N(v_k^{'}) = \left\{v_k, v_k^{''}\right\}$ and $N(v_k^{''}) = \left\{v_k, v_k^{'}\right\}$.

The notions of duplication of a vertex by a new edge and duplication of an edge by a new vertex were introduced by Vaidya and Barasara [5].

Definition 2.8: A *Book graph $B_{m,n}$* is a graph obtained by identifying n edges taking one edge from each of the n distinct copies of C_m, where n is called number of pages of the book $B_{m,n}$.

The edge obtained by identifying n edges is called the *spine* or *base* of the graph $B_{m,n}$.

Definition 2.9: The graph $B_{3,n}$ is called a *triangular book graph.*

2 Main Results

Theorem 2.1 *The graph G' obtained from a product cordial graph G by duplicating each of the vertices with an edge is a product cordial graph if G has even number of vertices.*

Proof: Let G be a product cordial graph with even number of vertices. Let $|V(G)| = n$ and $|E(G)| = m$. Let $f: V(G) \to \{0,1\}$ be a corresponding product cordial labeling such that $\left|v_f(0) - v_f(1)\right| \leq 1$ and $\left|e_f(0) - e_f(1)\right| \leq 1$. Clearly $v_f(0) = v_f(1) = \dfrac{n}{2}$. Let G' be the new graph obtained from G by duplicating each of the vertices by an edge. Thus $\left|V(G')\right| = n + 2n = 3n$ and $\left|E(G')\right| = m + 3n$. G' is constructed from G by attaching a triangle at each of the vertices of G.

- Case 1: m is even.
  Clearly, $e_f(0) = e_f(1) = \dfrac{m}{2}$. Our aim is to define $f': V(G') \to \{0,1\}$ such that f' admits product cordial labeling.

$$f'(x) = \begin{cases} f(x) & \text{, if } x \in V(G); \\ 1 & \text{, if } x \text{ is adjacent to } y, x \notin V(G), y \in V(G) \text{ and } f(y) = 1; \\ 0 & \text{, if } x \text{ is adjacent to } y, x \notin V(G), y \in V(G) \text{ and } f(y) = 0. \end{cases}$$

  Thus $v_{f'}(0) = v_{f'}(1) = \dfrac{3n}{2}$ and $e_{f'}(0) = e_{f'}(1) = \dfrac{m+3n}{2}$, which clearly preserves $\left|v_{f'}(0) - v_{f'}(1)\right| \leq 1$ and $\left|e_{f'}(0) - e_{f'}(1)\right| \leq 1$.

- Case 2: m is odd.
  Clearly, $e_f(0) = e_f(1) \pm 1$. Our aim is to define $f': V(G') \to \{0,1\}$ such that f' admits product cordial labeling.

$$f'(x) = \begin{cases} f(x) & \text{, if } x \in V(G); \\ 1 & \text{, if } x \text{ is adjacent to } y, x \notin V(G), y \in V(G) \text{ and } f(y) = 1; \\ 0 & \text{, if } x \text{ is adjacent to } y, x \notin V(G), y \in V(G) \text{ and } f(y) = 0. \end{cases}$$

Thus equal number of new edges are generated with label 0 and 1. Thus difference remains same, which clearly preserves $\left|v_{f'}(0) - v_{f'}(1)\right| \leq 1$ and $\left|e_{f'}(0) - e_{f'}(1)\right| \leq 1$.

Clearly, from both the cases G' admits product cordial labeling. Hence, G' is a product cordial graph.

Theorem 2.2 *The graph G' obtained from a product cordial graph G by duplicating each of the vertices with an edge is a product cordial graph if $|V(G)|$ is odd and $|E(G)|$ is even.*

Proof: Let G be a product cordial graph with $|V(G)| = n$ is odd and $|E(G)| = m$ is even. Let $f : V(G) \to \{0,1\}$ be corresponding product cordial labeling such that $\left|v_f(1) - v_f(0)\right| \leq 1$ and $\left|e_f(1) - e_f(0)\right| \leq 1$. Let G' be the new graph obtained from G by duplicating each of the vertices by an edge. Thus $\left|V(G')\right| = n + 2n = 3n$ and $\left|E(G')\right| = m + 3n$. G' is constructed from G by attaching a triangle at each of the vertices of G.

- Case 1: $v_f(1) - v_f(0) = 1$.
  Let $V_f(1) = \{u_1, u_2, u_3, \ldots, u_k, u_{k+1}\}$ and $V_f(0) = \{v_1, v_2, v_3, \ldots, v_k\}$ in G where $2k+1 = n$. This implies $|V_f(1)| = \dfrac{n+1}{2}$ and $|V_f(0)| = \dfrac{n-1}{2}$. Also $e_f(0) = e_f(1) = \dfrac{m}{2}$.
  $V(G') = V(G) \bigcup \left\{u_i^l, u_i^r / i = 1,2,\ldots,k+1\right\} \bigcup \left\{v_i^l, v_i^r / i = 1,2,\ldots,k\right\}$ and
  $E(G') = E(G) \bigcup \left\{u_i^l u_i, u_i^r u_i, u_i^l u_i^r / i = 1,2,\ldots,k+1\right\} \bigcup \left\{v_i^l v_i, v_i^r v_i, v_i^l v_i^r / i = 1,2,\ldots,k\right\}$.
  Our aim is to define $f' : V(G') \to \{0,1\}$ such that f' admits product cordial labeling.

$$f'(x) = \begin{cases} f(x) & \text{, if } x \in V(G); \\ 1 & \text{, if } x \in \left\{u_i^l, u_i^r / i = 1,2,\ldots,k\right\}; \\ 1 & \text{, if } x = u_{k+1}^l; \\ 0 & \text{, if } x = u_{k+1}^r; \\ 0 & \text{, if } x \in \left\{v_i^l, v_i^r / i = 1,2,\ldots,k\right\}. \end{cases}$$

  We get $v_{f'}(1) = \dfrac{n-1}{2} + n + 1$, $v_{f'}(0) = \dfrac{n+1}{2} + n - 1$, $e_{f'}(0) = \dfrac{m}{2} + \dfrac{3n+1}{2}$ and $e_{f'}(1) = \dfrac{m}{2} + \dfrac{3n-1}{2}$. Thus $\left|v_{f'}(0) - v_{f'}(1)\right| \leq 1$ and $\left|e_{f'}(0) - e_{f'}(1)\right| \leq 1$.

- Case 2: $v_f(0) - v_f(1) = 1$.
  Let $V_f(1) = \{u_1, u_2, u_3, \ldots, u_k\}$ and $V_f(0) = \{v_1, v_2, v_3, \ldots, v_k, v_{k+1}\}$ in G where $2k+1 = n$. This implies $|V_f(0)| = \dfrac{n+1}{2}$ and $|V_f(1)| = \dfrac{n-1}{2}$. Also $e_f(0) = e_f(1) = \dfrac{m}{2}$.
  $V(G') = V(G) \bigcup \left\{u_i^l, u_i^r / i = 1,2,\ldots,k\right\} \bigcup \left\{v_i^l, v_i^r / i = 1,2,\ldots,k+1\right\}$ and
  $E(G') = E(G) \bigcup \left\{u_i^l u_i, u_i^r u_i, u_i^l u_i^r / i = 1,2,\ldots,k\right\} \bigcup \left\{v_i^l v_i, v_i^r v_i, v_i^l v_i^r / i = 1,2,\ldots,k+1\right\}$.
  Our aim is to define $f' : V(G') \to \{0,1\}$ such that f' admits product cordial labeling.

$$f'(x) = \begin{cases} f(x) & \text{, if } x \in V(G); \\ 1 & \text{, if } x \in \left\{u_i^l, u_i^r / i = 1,2,\ldots,k\right\}; \\ 1 & \text{, if } x \in \left\{v_{k+1}^l, v_{k+1}^r\right\}; \\ 0 & \text{, if } x \in \left\{v_i^l, v_i^r / i = 1,2,\ldots,k\right\}. \end{cases}$$

  We get $v_{f'}(1) = \dfrac{n-1}{2} + n + 1$, $v_{f'}(0) = \dfrac{n+1}{2} + n - 1$, $e_{f'}(0) = \dfrac{m}{2} + \dfrac{3n+1}{2}$ and $e_{f'}(1) = \dfrac{m}{2} + \dfrac{3n-1}{2}$. Thus $\left|v_{f'}(0) - v_{f'}(1)\right| \leq 1$ and $\left|e_{f'}(0) - e_{f'}(1)\right| \leq 1$.

Clearly from both the cases G' admits product cordial labeling. Hence, G' is a product cordial graph.

Theorem 2.3 *Book graph $B_{m,n}$ is a product cordial graph if both m and n are odd, for $m \geq 3$.*

Proof: Let $G = B_{m,n}$ be a book graph obtained by identifying n edges taking one edge from each of the n distinct copies of C_m, where n is called number of pages of the book. The edge obtained by identifying n edges is called the spine.
Thus $|V(G)| = (m-2)n+2$ and $|E(G)| = (m-1)n+1$.
We name the vertices of G as follows: v_0 and v_0' to the end vertices of the spine in $B_{m,n}$. Let $v_1, v_2, v_3, \ldots, v_{(m-2)n}$ be the consecutive vertices of 1^{st} copy of C_m, 2^{nd} copy of $C_m, \ldots, n^{th}$ copy of C_m other than the end vertices of the spine.
Define a function $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & , \text{if } x \in \{v_0, v_0'\}; \\ 1 & , \text{if } x = v_i, \ 1 \le i \le \dfrac{(m-2)(n-1)}{2} + \dfrac{m-3}{2}; \\ 0 & , \text{if } x = v_i, \ \dfrac{(m-2)(n-1)}{2} + \dfrac{m-3}{2} < i \le (m-2)n. \end{cases}$$

Thus we get $v_f(0) = \dfrac{(m-2)n+1}{2}$, $v_f(1) = \dfrac{(m-2)n+3}{2}$, $e_f(0) = \left\lceil \dfrac{(m-1)n+1}{2} \right\rceil$ and $e_f(1) = \left\lfloor \dfrac{(m-1)n+1}{2} \right\rfloor$. Thus $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Hence, G admits product cordial labeling. G is a product cordial graph when both m and n are odd and $m \ge 4$.

Illustration 2.1 *Book graph $B_{5,3}$ is a product cordial graph.*

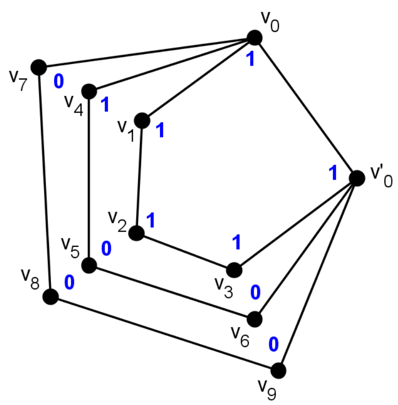

Figure 1: $m = 5$ and $n = 3$

Theorem 2.4 *Book graph $B_{m,n}$ is not product cordial graph if either of m or n is even, for $m \ge 3$.*

Proof: Let $G = B_{m,n}$ be a book graph obtained by identifying n edges taking one edge from each of the n distinct copies of C_m, where n is called number of pages of the book. The edge obtained by identifying n edges is called the spine.
Thus $|V(G)| = (m-2)n+2$ and $|E(G)| = (m-1)n+1$.
We name the vertices of G as follows: v_0 and v_0' to the end vertices of the spine in $B_{m,n}$. Let $v_1, v_2, v_3, \ldots, v_{(m-2)n}$ be the consecutive vertices of 1^{st} copy of C_m, 2^{nd} copy of $C_m, \ldots, n^{th}$ copy of C_m other than the end vertices of the spine.

- Case 1: Both m and n are even.
  We have to label $\dfrac{(m-2)n+2}{2}$ vertices with label 0 and $\dfrac{(m-2)n+2}{2}$ vertices with label 1 to satisfy the condition of product cordial labeling. Thus $v_f(0) = v_f(1) = \dfrac{(m-2)n+2}{2}$. So we will get atleast $\dfrac{(m-1)n}{2} + 2$

edges with label 0 and atmost $\frac{(m-1)n}{2}-1$ edges with label 1.
Hence $|e_f(1)-e_f(0)|=3>1$.

- Case 2: m is even and n is odd.
  We have to label $\frac{(m-2)n+2}{2}$ vertices with label 0 and $\frac{(m-2)n+2}{2}$ vertices with label 1 to satisfy the condition of product cordial labeling. Thus $v_f(0)=v_f(1)=\frac{(m-2)n+2}{2}$. So we will get atleast $\frac{(m-1)n+3}{2}$ edges with label 0 and atmost $\frac{(m-1)n-1}{2}$ edges with label 1.
  Hence $|e_f(1)-e_f(0)|=2>1$.

- Case 3: m is odd and n is even.
  We have to label $\frac{(m-2)n+2}{2}$ vertices with label 0 and $\frac{(m-2)n+2}{2}$ vertices with label 1 to satisfy the condition of product cordial labeling. Thus $v_f(0)=v_f(1)=\frac{(m-2)n+2}{2}$. So we will get atleast $\frac{(m-1)n}{2}+2$ edges with label 0 and atmost $\frac{(m-1)n}{2}-1$ edges with label 1.
  Hence $|e_f(1)-e_f(0)|=3>1$.

Thus from all the three cases G does not admit product cordial labeling.
Hence, G is not a product cordial graph.

Corollary 2.1 *Triangular book graph $B_{3,n}$ is a product cordial graph if n is odd.*

Proof: Using both the above theorems 3.3 and 3.4 we can prove that triangular book graph is product cordial graph if n is odd.

Illustration 2.2 *$B_{3,3}$ is a product cordial graph.*

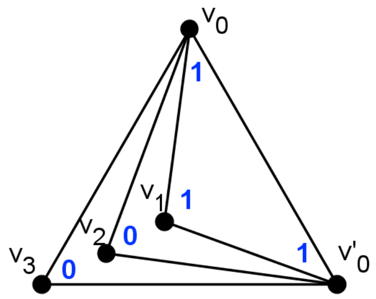

Figure 2: $m=3$ and $n=3$

Theorem 2.5 *The graph obtained by duplication of the spine with a vertex in book graph $B_{m,n}$ is a product cordial graph for $m\geq 4$.*

Proof: Let $B_{m,n}$ be a book graph obtained by identifying n edges taking one edge from each of the n distinct copies of C_m, where n is called number of pages of the book. The edge obtained by identifying n edges is called the spine.
Thus $|V(B_{m,n})|=(m-2)n+2$ and $|E(B_{m,n})|=(m-1)n+1$.
We name the vertices of $B_{m,n}$ as follows: v_0 and v'_0 to the end vertices of the spine in $B_{m,n}$. Let $v_1,v_2,v_3,\ldots,v_{(m-2)n}$ be the consecutive vertices of 1^{st} copy of C_m, 2^{nd} copy of $C_m,\ldots,n^{th}$ copy of C_m other than the end vertices of the spine.

Let G be the graph obtained from $B_{m,n}$ by duplicating spine with a vertex v_s we get $|V(G)| = (m-2)n+3$ and $|E(G)| = (m-1)n+3$. Define a function $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{, if } x \in \{v_0,\ v_0',\ v_s\}; \\ 1 & \text{, if } x = v_i,\ 1 \le i \le \left\lfloor \dfrac{(m-2)n-2}{2} \right\rfloor; \\ 0 & \text{, if } x = v_i,\ \left\lfloor \dfrac{(m-2)n-2}{2} \right\rfloor < i \le (m-2)n. \end{cases}$$

Thus we get $e_f(0) = \left\lceil \dfrac{(m-1)n+3}{2} \right\rceil$ and $e_f(1) = \left\lfloor \dfrac{(m-1)n+3}{2} \right\rfloor$.
Thus $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.
Thus, G admits product cordial labeling. Hence, G is a product cordial graph.

Illustration 2.3 *The graph obtained by duplication of spine with a vertex in $B_{6,2}$ and $B_{7,2}$ is product cordial graph.*

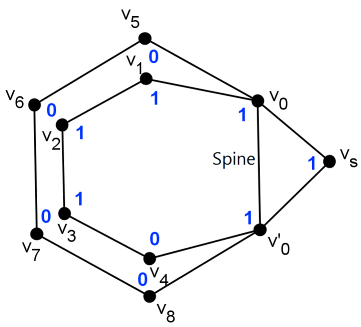

Figure 3: $m = 6$ and $n = 2$

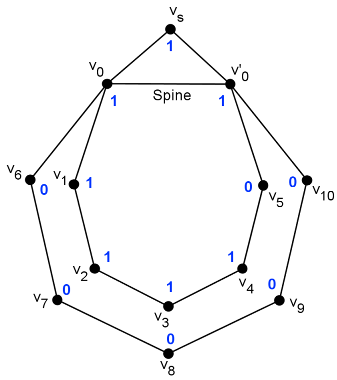

Figure 4: $m = 7$ and $n = 2$

Theorem 2.6 *The graph obtained from a triangular book graph $B_{3,n}$ by duplicating each of the vertices with an edge is a product cordial graph.*

Proof: Let $G = B_{3,n}$ be a triangular book graph obtained by identifying n edges taking one edge from each of the n distinct copies of C_3, where n is called number of pages of the book. The edge obtained by identifying n edges is called the spine.
Thus $|V(G)| = n+2$ and $|E(G)| = 2n+1$.
We name the vertices of $B_{3,n}$ as follows: v_0 and $v_0^{'}$ to the end vertices of the spine. Let $v_1, v_2, v_3, \ldots, v_n$ be the consecutive vertices of $1^{st}, 2^{nd}, \ldots, n^{th}$ copy of C_3 other than the end vertices of the spine.
Let G' be the graph obtained from G by duplicating each of the vertex by an edge. Thus $|V(G')| = 3(n+2)$ and $|E(G')| = 5n+7$.
$V(G') = V(G) \bigcup \left\{v_0^r, v_0^l, v'^l_0, v'^r_0\right\} \bigcup \left\{v_i^l, v_i^r / i = 1,2,\ldots,n\right\}$ and
$E(G') = E(G) \bigcup \left\{v'_0 v'^l_0, v'_0 v'^r_0, v'^l_0 v'^r_0\right\} \bigcup \left\{v_i^l v_i, v_i^r v_i, v_i^l v_i^r / i = 0,1,2,\ldots,n\right\}$. Define $f : V(G') \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{, if } x \in V(G); \\ 1 & \text{, if } x = v_0^l; \\ 1 & \text{, if } x = v_i^r, 0 \le i \le \left\lfloor \dfrac{n-1}{2} \right\rfloor; \\ 0 & \text{, otherwise.} \end{cases}$$

Thus we get $v_f(0) = \left\lfloor \dfrac{3n+6}{2} \right\rfloor$, $v_f(1) = \left\lceil \dfrac{3n+6}{2} \right\rceil$, $e_f(0) = \left\lceil \dfrac{5n+7}{2} \right\rceil$ and $e_f(1) = \left\lfloor \dfrac{5n+7}{2} \right\rfloor$. Thus $\left|e_f(0) - e_f(1)\right| \leq 1$ and $\left|v_f(0) - v_f(1)\right| \leq 1$. Thus G' admits product cordial labeling. Hence G' is a product cordial graph.

Illustration 2.4 *The graph obtained by duplication of all the vertices with an edge in $B_{3,1}$ is product cordial graph.*

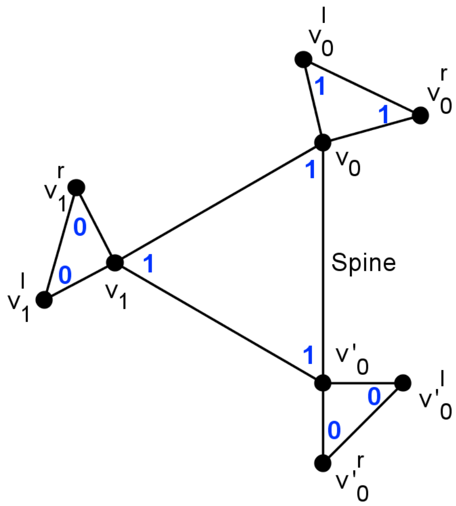

Figure 5: $m = 3$ and $n = 1$

Theorem 2.7 *An edge deleted subgraph obtained from the graph $B_{m,n}$ by deleting all the edges incident on one of the end vertices of spine in book graph admits product cordial labeling.*

Proof: Let $G = B_{m,n}$ be a book graph obtained by identifying n edges taking one edge from each of the n distinct copies of C_m, where n is called number of pages of the book. The edge obtained by identifying n edges is called the spine.
Thus $|V(G)| = (m-2)n+2$ and $|E(G)| = (m-1)n+1$.
We name the vertices of G as follows: v_0 and v'_0 to the end vertices of the spine in $B_{m,n}$. Let $v_{1k}, v_{2k}, v_{3k}, \ldots, v_{(m-2)k}$, for $k \in \{1,2,\ldots n\}$ be the consecutive vertices of k^{th} copy of C_m other than the end vertices of the spine.
Let G' be the edge deleted subgraph obtained from G by deleting all the edges incident on v_0. Then we have $|V(G')| = |V(G)|$ and $|E(G')| = |E(G)| - d(v_0) = (m-2)n$.

- Case 1: m is even.
  Define $f : V(G') \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{, if } x = v_0; \\ 1 & \text{, if } x = v'_0; \\ 0 & \text{, if } x = v_{ij},\ i = 1,2,3\ldots n \text{ and } 1 \leq j \leq \dfrac{m-2}{2}; \\ 1 & \text{, if } x = v_{ij},\ i = 1,2,3\ldots n \text{ and } \dfrac{m-2}{2} < j \leq n. \end{cases}$$

  Thus we get $v_f(0) = v_f(1) = \dfrac{(m-2)n+2}{2}$, $e_f(0) = e_f(1) = \dfrac{(m-2)n}{2}$.
  Thus $\left|e_f(0) - e_f(1)\right| \leq 1$ and $\left|v_f(0) - v_f(1)\right| \leq 1$.

- Case 2: m is odd.
  Define $f : V(G') \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{, if } x = v_0; \\ 0 & \text{, if } x = v_{ij},\ 1 \le j \le \dfrac{m-3}{2} \text{ and } i = 1,2,\dots,n; \\ 0 & \text{, if } x = v_{ij},\ j = \dfrac{m-1}{2} \text{ and } 1 \le i \le \left\lfloor \dfrac{n-1}{2} \right\rfloor; \\ 1 & \text{, if } x = v'_0; \\ 1 & \text{, if } x = v_{ij}, \text{ otherwise.} \end{cases}$$

  Thus $v_f(0) = \left\lfloor \dfrac{(m-2)n+2}{2} \right\rfloor$, $v_f(1) = \left\lceil \dfrac{(m-2)n+2}{2} \right\rceil$, $e_f(0) = \left\lfloor \dfrac{(m-2)n}{2} \right\rfloor$ and $e_f(1) = \left\lceil \dfrac{(m-2)n}{2} \right\rceil$.
  Thus $|e_f(0) - e_f(1)| \le 1$ and $|v_f(0) - v_f(1)| \le 1$.

From both the cases G' admits product cordial labeling. Thus, it is product cordial graph.

Illustration 2.5 *The graph obtained by deleting all the edges incident on v_0 in $B_{4,3}$ is product cordial graph.*

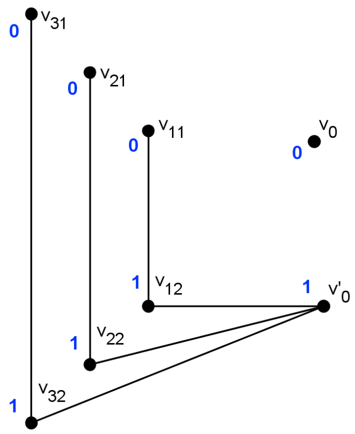

Figure 6: $m = 4$ and $n = 3$

Conclusion

Here we have proved that the graph obtained by duplication of every vertex in product cordial graph is a product cordial graph if the graph has even number of vertices. Also the graph obtained by duplication of every vertex in product cordial graph if the cardinality of vertex set is odd and the cardinality of edge set is even, is product cordial graph. We have also proved book graph is product cordial graph if and only if both edge set and vertex set are odd. Various results based on book graph are proved using some of the graph operations.

Problem 1 *Does the graph obtained by duplication of every vertex in product cordial graph, if the cardinalities of both the vertex set and the edge set are odd, admits product cordial labeling?*

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