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Solution of One Dimensional Telephone Equation Using Method of Separation of Variables

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Abstract

In this paper one dimensional second order hyperbolic telegraph equation has been obtained by using Ohm's law and electric circuit. The closed approximate solution of governing partial differential equation has been obtained by using method of separation of variables which can be used in study of wave propagation of electrical signal in cable transmission line.

Keywords: Telegraph equation, Ohm’s law, Electric circuit, Separation of variables, Cable transmission line, and Wave propagation

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1. Introduction:

In this modern era, communication system plays an important role in science and technology. This system use the transmission media for transferring the information carrying signal form one point to another point, this transmission media can be categorized into two groups namely guided and unguided. In guided media the signal is transferred through the co-axial cable or transmission line, where as in unguided media electromagnetic wave are transmitted and received through antenna.

Electronic circuit consists of different components such as Inductor, Capacitor and Resistor etc. Therefore there is fluctuation in the voltage and current, which can be determined by formulating some equations. Telegraph equation is one of those equations.

A telegraph equation is a pair of coupled linear differential equations that describe the voltage and current on an electrical transmission line with distance and time. This equation has been obtained by Oliver Heaviside (1880) who developed the transmission line model, which demonstrate that the electromagnetic wave can be reflected on the wire and that wave patterns can appears along the line. The theory applies to transmission line of all frequency including high frequency transmission lines (such as telegraph wires and radio frequency conductors), audio frequency (such as telephone lines) and direct current. Telephone equation is commonly used in study of wave propagation of electric signal in a cable transmission line and also in wave phenomena [4].

2. Mathematical Formulation

Consider an infinitesimal piece of telephone cable. The electrical Circuit is shown in given Figure 1. Further assuming that the cable is imperfectly insulated so that there are both capacitance and current leakage to ground.

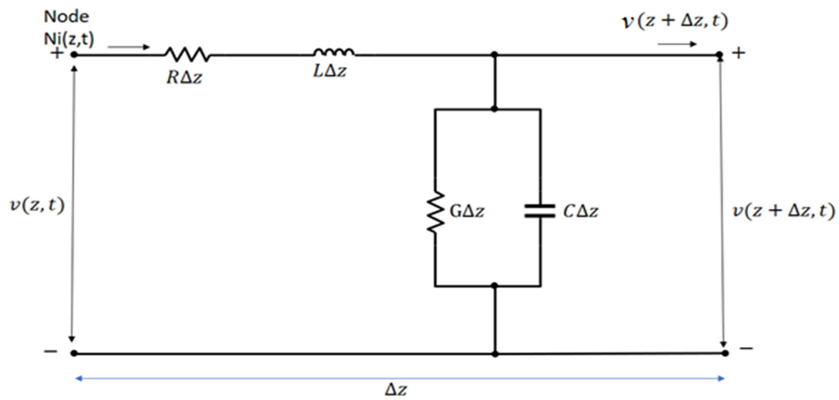

Figure 1

Δz = length of the cable (m)

$v(z,t)$= voltage at any point and any time on the cable

$i(z,t)$= current at any point and any time on the cable

R = resistance (ohm) per unit length

L = inductance (henry) per unit length

C = capacitance (farad) per unit length

G = conductance (mho/m) per unit length

Transmission line equation is formulated using Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL).

Using Kirchhoff's Voltage Law,

$$v(z,t) - R\Delta z i - L\Delta z\frac{di}{dt} - v(z+\Delta z,t) = 0 \tag{2.1}$$

Dividing by Δz, we get

$$-\left[\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z}\right] = Ri + L\frac{di}{dt}$$

Taking the limit $\Delta z \to 0$, and differentiating with respect to z, we have

$$\frac{dv}{dz} = -Ri - L\frac{di}{dt} \tag{2.2}$$

Now, using Kirchhoff's Current Law, we get

$$i(z,t) = i(z+\Delta z,t) + G\Delta z v(z+\Delta z,t) - i_c\Delta z$$

$$Now, i_c = \frac{dq}{dt}\ and\ q = Cv \Longrightarrow \frac{dq}{dt} = C\frac{dv}{dt}$$

$$\therefore i(z,t) = i(z+\Delta z,t) + G\Delta z v(z+\Delta z,t) + C\Delta z\frac{dv}{dt}(z+\Delta z,t) \tag{2.3}$$

Dividing by Δz, we get

$$-\left[\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z}\right] = Gv + C\frac{dv}{dt}$$

Taking the limit $\Delta z \to 0$, and differentiating with respect to z, we have

$$\frac{di}{dz} = -Gv - C\frac{dv}{dt} \tag{2.4}$$

Now considering equation (2.2) in the following form

$$\left(R + L\frac{d}{dt}\right)i + \frac{dv}{dz} = 0$$

Taking derivative with respect to z, we get

$$\left(R + L\frac{d}{dt}\right)\frac{di}{dz} + \frac{d^2v}{dz^2} = 0 \tag{2.5}$$

From equation (2.4) we get,

$$\frac{di}{dz} + \left(G + C\frac{d}{dt}\right)v = 0$$

Multiplying above equation by $\left(R + L\frac{d}{dt}\right)$

$$\left(R + L\frac{d}{dt}\right)\frac{di}{dz} + \left(R + L\frac{d}{dt}\right)\left(G + C\frac{d}{dt}\right)v = 0 \tag{2.6}$$

Subtracting equation (2.6) from (2.5), we get

$$\frac{d^2v}{dz^2} - \left(G + L\frac{d}{dt}\right)\left(G + C\frac{d}{dt}\right)v = 0$$

On simplifying, we get

$$\frac{d^2v}{dz^2} = LC\frac{d^2v}{dz^2} + (RC + GL)\frac{dv}{dt} + RGv$$

$$\frac{d^2v}{dz^2} = LC\left[\frac{d^2v}{dz^2} + \left(\frac{G}{C} + \frac{R}{L}\right)\frac{dv}{dt} + \frac{RG}{LC}v\right]$$

This can be written as

$$\frac{1}{LC}\frac{d^2v}{dz^2} = \frac{d^2v}{dt^2} + \left(\frac{G}{C} + \frac{R}{L}\right)\frac{dv}{dt} + \frac{RG}{LC}v \tag{2.7}$$

Substituting $\frac{G}{C} = \alpha,\ \ \frac{R}{L} = \beta,\ \ \frac{1}{LC} = c^2$, we get

$$c^2\frac{d^2v}{dz^2} = \frac{d^2v}{dt^2} + (\alpha + \beta)\frac{dv}{dt} + (\alpha\beta)v \quad , \quad \text{where, } \alpha, \beta, c \text{ are arbitrary constants} \tag{2.8}$$

This is known as one-dimensional second order hyperbolic telephone equation for voltage.

Proceeding in the similar way, we get

$$c^2\frac{d^2i}{dz^2} = \frac{d^2i}{dt^2} + (\alpha + \beta)\frac{di}{dt} + (\alpha\beta)i \tag{2.9}$$

This is known as one-dimensional second order hyperbolic telephone equation for current.

Equation (2.8) and (2.9) together are known as Telephone equations [5].

3. Solution of the problem

Telephone equations (2.8) and (2.9) can be solved by using different methods like numerical method, sine collection method, reduced differential transform method and separation of variables method.

Here we are use method of separation of variables to obtain the solution

$$c^2\, v_{zz} = v_{tt} + (\alpha + \beta)v_t + (\alpha\beta)v \tag{3.1}$$

Now we solve boundary value problem with following conditions:[6]

Boundary condition: $v(0,t) = 0$ (3.2)

$v(l,t) = 0$ (3.3)

Initial condition: $v(z,0) = f(z)$ (3.4)

$$\frac{dv(z,0)}{dt} = 0 \quad (3.5)$$

Let, the solution of equation (3.1) be expressed as

$$v(z,t) = Z(z) \cdot T(t) \quad (3.6)$$

From equation (3.1) and (3.6), we have

$\therefore ZT''+(\alpha+\beta)ZT'+(\alpha\beta)ZT=c^2Z''T$

$$\therefore \frac{1}{c^2}\left[\frac{T''}{T} + (\alpha+\beta)\frac{T'}{T} + (\alpha\beta)\right] = \frac{Z''}{Z}$$

This equality holds only when each term compare with constant K.

$$\therefore \frac{T''}{T} + (\alpha+\beta)\frac{T'}{T} + (\alpha\beta) = c^2K, \quad Z'' = Kz \quad (3.7)$$

Now, we will consider three possibilities for K,

(1) $K > 0$, (2) $K = 0$, (3) $K < 0$

Case 1: When $K > 0$, the solution of equation (3.7) is

$$v(z,t) = (c_1e^{Pz} + C_2e^{-Pz})(C_3e^{r_1t} + C_4e^{r_2t})$$

where, $K = P^2$ and $r_j = \dfrac{-(\alpha+\beta) \pm \sqrt{(\alpha-\beta)^2 + (2PC)^2}}{2} \;; j = 1,2$

Case 2: When $K = 0$, the solution of equation (3.7) is

$$v(z,t) = (C_1z + C_2)(C_3e^{r_1t} + C_4e^{r_2t})$$

where, $K = 0$ and $r_j = \dfrac{-(\alpha+\beta) \pm (\alpha-\beta)}{2}; j = 1,2$

Case 3: $when$ $K < 0$, The solution of equation (3.7) is

$$v(z,t) = (C_1cosPz + C_2sinPz)(C_3e^{r_1t} + C_4e^{r_2t})$$

where, $K = -P^2$ and $r_j = \dfrac{-(\alpha+\beta) \pm \sqrt{(\alpha-\beta)^2 - (2PC)^2}}{2}; j = 1,2$

As the physical nature of this problem is periodic, we will consider Case 3 as appropriate solution.

Therefore, the general solution is

$$v(z,t) = (C_1cosPz + C_2sinPz)(C_3e^{r_1t} + C_4e^{r_2t})$$

$$\text{where, } r_j = \frac{-(\alpha+\beta) \pm \sqrt{(\alpha-\beta)^2 - (2PC)^2}}{2}; j = 1,2 \quad (3.8)$$

Taking $\gamma = \dfrac{(\alpha+\beta)}{2}$,

$$4{\omega_n}^2 = -\{(\alpha-\beta)^2 - 4P^2C^2\} \Rightarrow i\omega_n = \sqrt{\frac{(\alpha-\beta)^2 - 4P^2C^2}{2}}$$

Therefore $r_j = -\gamma \pm i\omega_n$

Substituting the value of r_j in equation (3.8), we get

$$T = C_3e^{(-\gamma+i\omega_n)t} + C_4e^{(-\gamma-i\omega_n)t}$$

$= e^{-\gamma t}[C_3(cos\omega_n t + isin\omega_n t) + C_4(cos\omega_n t - isin\omega_n t)]$

$= e^{-\gamma}\ [Acos\omega_n t + Bsin\omega_n t]$, where $A = C_3 + C_4\,, B = C_3 - iC_4$

Therefore, $v(z,t) = (C_1\cos Pz + C_2 sinPz)\cdot e^{-\gamma t}(Acos\omega_n t + Bsin\omega_n t)$ (3.9)

Now, using boundary condition (3.2) in equation (3.9), we get

$C_1 e^{-\gamma t}(Acos\omega_n t + Bsin\omega_n t) = 0 \Rightarrow C_1 = 0$

Therefore equation (3.9) reduces to

$v = C_2 sinPze^{-\gamma t}(Acos\omega_n t + Bsin\omega_n t)$ (3.10)

Now, using another boundary condition (3.3) in equation (3.10), we get

$v = C_2 sinPle^{-\gamma t}(Acos\omega_n t + Bsin\omega_n t)$ where, $Acos\omega_n t + Bsin\omega_n t \neq 0$

Therefore, $C_2 sinPl = 0$

This gives, $P = \frac{n\pi}{l}$

Equation (3.10) reduces to,

$$v = C_2 sin\frac{n\pi z}{l}e^{-\gamma t}[Acos\omega_n t + Bsin\omega_n t]$$

By considaring $B = sin\emptyset_n$, we get $v = C_2 sin\frac{n\pi z}{l}e^{-\gamma}\ cos(\omega_n t - \emptyset_n)$

Therefore the general solution can be written as,

$$v(z,t) = \sum_{n=1}^{\infty} A_n sin\frac{n\pi z}{l}e^{-\gamma t}\, cos(\omega_n t - \emptyset_n) \qquad (3.11)$$

Now, using initial condition (3.5) we have,

$$\left(\frac{dv}{dt}\right)_n = \sum_{n=1}^{\infty} A_n[-\gamma e^{-\gamma t}cos(\omega_n t - \emptyset_n) - w_n e^{-\gamma t}sin(\omega_n t - \emptyset_n)]\cdot sin\left(\frac{n\pi z}{l}\right)$$

$$v_t(z,0) = \sum_{n=1}^{\infty} A_n[-\gamma cos\emptyset_n + \omega_n sin\emptyset_n]\cdot sin\left(\frac{n\pi z}{l}\right) = 0$$

This gives

$$-\gamma cos\emptyset_n + \omega_n sin\emptyset_n = 0 \Rightarrow \emptyset_n = tan^{-1}\left(\frac{\gamma}{\omega_n}\right)$$

Now, using another initial condition (3.4), Equation (3.11) reduces to,

$$f(z) = \sum_{n=1}^{\infty} A_n sin\frac{n\pi z}{l}e^{-\gamma t}\cos(\emptyset_n)\ ,$$ where, $\emptyset_n$ is an arbitrary phase.

The final solution is,

$$v(z,t) = \sum_{n=1}^{\infty} A_n e^{-\gamma t}\, cos(\omega_n t - \emptyset_n)\, sin\left(\frac{n\pi z}{l}\right) \qquad (3.12)$$

where

$$\gamma = \frac{(\alpha+\beta)}{2}, \qquad \omega_n = \sqrt{\left(\frac{n\pi c}{l}\right)^2 - \frac{1}{4}(\alpha-\beta)^2}\ , \qquad \emptyset_n = tan^{-1}\left(\frac{\gamma}{\omega_n}\right),$$

$$A_n = \frac{2}{lcos\emptyset_n}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)\, dx$$

To interpret the result, rewriting the solution obtained in (3.12) as

$$v(z,t) = \sum_{n=1}^{\infty} \frac{1}{2} A_n e^{-\gamma t} \left[sin\left(\frac{n\pi z}{l} - \omega_n t + \emptyset_n\right) + sin\left(\frac{n\pi z}{l} + \omega_n t - \emptyset_n\right) \right] \quad (3.13)$$

If we carefully tune the wire so that $\alpha = \beta$ then $\gamma = \alpha$, $\omega_n = \frac{n\pi c}{l}$

Therefore, we get

$$v(z,t) = \sum_{n=1}^{\infty} \frac{1}{2} A_n e^{-\alpha t} \left[sin\left(\frac{n\pi z}{l} - \frac{n\pi ct}{l} + \emptyset_n\right) + sin\left(\frac{n\pi z}{l} + \frac{n\pi ct}{l} - \emptyset_n\right) \right]$$

This can be written as

$$v(z,t) = e^{-\alpha t} f(z - ct) + e^{-\alpha t} g(z + ct) \quad (3.14)$$

where, $f(x) = \sum_{n=1}^{\infty} \frac{1}{2} A_n \, sin\left(\frac{n\pi x}{l} + \emptyset_n\right)$ *and* $g(x) = \sum_{n=1}^{\infty} \frac{1}{2} A_n \, sin\left(\frac{n\pi x}{l} - \emptyset_n\right)$

4. Conclusion

The solution of one-dimensional second order hyperbolic telephone equation (2.8) is obtained in the form of Fourier series in equation (3.14). To conclude, let us consider the three cases: (i) When α=β>0, the function v(z,t) represents the summation of two signals which propagates in opposite direction without changing the shape. (ii) When α=β=0 then no signal propagates in any direction. (iii) When α≠β then different frequency component of the function v(z,t) propagate with different speed as ω_n depends on n and the signals get distorted. This phenomenon is known as dispersion.

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