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CHARACTERIZATION OF GELFAND SPACE OF THE BANACH ALGEBRA $\mathcal{A} \times_d \mathcal{B}$ WITH DIRECT-SUM PRODUCT

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Abstract. Let $\mathcal{A}$ be a semisimple commutative Banach algebra and $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$. A multiplication, generalizing the product defined on $\mathcal{A}_e$, is defined on the cartesian product $\mathcal{A} \times \mathcal{B}$. Which gives a new Banach algebra $\mathcal{A} \times_d \mathcal{B}$. The Gelfand space and the Shilov boundary of $\mathcal{A} \times_d \mathcal{B}$ is characterized in terms of that of $\mathcal{A}$ and $\mathcal{B}$. Using that uniqueness properties and regularity of $\mathcal{A} \times_d \mathcal{B}$ is also discussed.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras. Then the Gelfand space of the cartesian product $\mathcal{A}\times\mathcal{B}$ is in terms of that of $\mathcal{A}$ and $\mathcal{B}$ is characterized in [3]. If $\mathcal{A}$ is a non-unital algebra, then $\mathcal{A}_e = \mathcal{A} \times \mathbb{C}$ becomes a unital algebra with the product defined as $(a,\alpha)(b,\beta) = (ab+\alpha b+\beta a, \alpha\beta)$ $((a,\alpha),(b,\beta) \in \mathcal{A}_e)$. Here, We generalize the product defined on $\mathcal{A}_e$. Let $\mathcal{A}$ be an algebra and $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Then $\mathcal{A} \times_d \mathcal{B}$ is an algebra with co-ordinatewise linear operations and the *direct-sum product* defined as

$$(a,b)(c,d) = (ac + ad + bc, bd) \quad ((a,b),(c,d) \in \mathcal{A} \times_d \mathcal{B}).$$

It is commutative (resp. unital) iff $\mathcal{A}$ is commutative (resp. unital). Further, If $\mathcal{A}$ is a normed algebra (resp. Banach algebra), then $\mathcal{A}\times_d\mathcal{B}$ is a normed algebra (resp. Banach algebra) with the norm $\|(a,x)\|_1 = \|a\| + \|x\|$ $((a,x) \in \mathcal{A} \times_d \mathcal{B})$.

Remark 1.1. Let $\mathcal{A}$ be non-unital and $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Though $\mathcal{A}\times_d\mathcal{B}$ has the same multiplication as in $\mathcal{A}_e$, it does not have identity even if $\mathcal{B}$ has identity.

2. Basic Properties

Throughout, let $\mathcal{A}$ be an algebra and $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Let $\mathcal{A}_{-1}$ denote the set of all quasi invertible elements of $\mathcal{A}$. If $\mathcal{A}$ is unital, $\mathcal{A}^{-1}$ is the set of all invertible elements of $\mathcal{A}$. Further, $\sigma_{\mathcal{A}}(a)$ and $r_{\mathcal{A}}(a)$ denote the spectrum and the spectral radius of a in $\mathcal{A}$. Then we have the following.

Proposition 2.1. *Let $(a,b) \in \mathcal{A} \times_d \mathcal{B}$. Then*

(1) $(a,b) \in (\mathcal{A} \times_d \mathcal{B})^{-1}$ *iff* $a+b \in \mathcal{A}^{-1}$ *and* $b \in \mathcal{B}^{-1}$;
(2) $(a,b) \in (\mathcal{A} \times_d \mathcal{B})_{-1}$ *iff* $a+b \in \mathcal{A}_{-1}$ *and* $b \in \mathcal{B}_{-1}$;
(3) $\sigma_{\mathcal{A}\times_d\mathcal{B}}((a,b)) = \sigma_{\mathcal{A}}(a+b) \cup \sigma_{\mathcal{B}}(b)$;
(4) $r_{\mathcal{A}\times_d\mathcal{B}}((a,b)) = \max\{r_{\mathcal{A}}(a+b), r_{\mathcal{B}}(b)\}$.

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Proposition 2.2. *Let $\mathcal{A}$ be a normed algebra and $\mathcal{B}$ be closed in $\mathcal{A}$. Then $\mathcal{A} \times_d \mathcal{B}$ has a left approximate identity iff $\mathcal{A}$ has a left approximate identity. (Similar results are true for right, bounded left, bounded right approximate identity)*

Proof. Suppose that $\mathcal{A} \times_d \mathcal{B}$ has left approximate identity $((e_\alpha, f_\alpha))$. Then it is easy to see that $(e_\alpha + f_\alpha)$ is a left approximate identity for $\mathcal{A}$ and (f_α) is a left approximate identity for $\mathcal{B}$.

Conversely, suppose that both $\mathcal{A}$ and $\mathcal{B}$ have left approximate identities. Let (e_α) and (f_β) be the left approximate identities for $\mathcal{A}$ and $\mathcal{B}$, respectively. If $(a, b) \in \mathcal{A} \times_d \mathcal{B}$, then

$$\begin{aligned}
&\|(e_\alpha - f_\beta, f_\beta)(a, b) - (a, b)\|_1 \\
&\quad = \|e_\alpha a + e_\alpha b - f_\beta b - a, f_\beta b - b\|_1 \\
&\quad = \|(e_\alpha a - a) + (e_\alpha b - b) + (b - f_\beta b)\| + \|f_\beta b - b\| \\
&\quad \leq \|e_\alpha a - a\| + \|e_\alpha b - b\| + 2\|b - f_\beta b\|
\end{aligned}$$

converges to 0 as $\alpha, \beta \to \infty$. Thus $(e_\alpha - f_\beta, f_\beta)$ is a left approximate identity for $\mathcal{A} \times_d \mathcal{B}$. Therefore $\mathcal{A} \times_d \mathcal{B}$ has left approximate identity. □

Remark 2.3. Let $\|\cdot\|$ be a norm on an algebra $\mathcal{A}$ and $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Let $\|(a, b)\|_\infty = \max\{\|a\|, \|b\|\}$ $((a, b) \in \mathcal{A} \times_d \mathcal{B})$. Then $\|\cdot\|_\infty$ may not be an algebra norm on $\mathcal{A} \times_d \mathcal{B}$.

Definition 2.4. [1, 2] Let $\mathcal{A}$ be an algebra. Then

(1) An algebra norm $||\cdot||$ on $\mathcal{A}$ is a *uniform norm* if $||a^2|| = ||a||^2$ $(a \in \mathcal{A})$.

(2) $\mathcal{A}$ is a *uniform* algebra if it admits a complete uniform norm.

(3) An algebra norm $||\cdot||$ on a $*$-algebra $\mathcal{A}$ is C^**-norm* if $||a^*a|| = ||a||^2$ $(a \in \mathcal{A})$.

Lemma 2.5. *Define* $|(a, b)| := \max\{\|a + b\|, \|b\|\}((a, b) \in \mathcal{A} \times_d \mathcal{B})$. *Then*

(1) $|\cdot|$ *is a norm on* $\mathcal{A} \times_d \mathcal{B}$*;*

(2) $|\cdot|$ *is a uniform norm on* $\mathcal{A} \times_d \mathcal{B}$ *iff* $\|\cdot\|$ *is a uniform norm on* $\mathcal{A}$*;*

(3) *Let* $\mathcal{A}$ *be a* $*$*-algebra and* $\mathcal{B}$ *be a* $*$*-subalgebra in* $\mathcal{A}$. *Then* $|\cdot|$ *is a* C^**-norm on* $\mathcal{A} \times_d \mathcal{B}$ *iff* $\|\cdot\|$ *is a* C^**-norm on* $\mathcal{A}$.

Corollary 2.6. *Let $\mathcal{B}$ be a closed subalgebra of a Banach algebra $\mathcal{A}$. Then $\mathcal{A} \times_d \mathcal{B}$ is a uniform algebra if and only if $\mathcal{A}$ is a uniform algebra.*

Proof. Since $\mathcal{A} \cong \mathcal{A} \times \{0\}$ is a closed subalgebra of $\mathcal{A} \times_d \mathcal{B}$, $\mathcal{A}$ is a uniform algebra whenever $\mathcal{A} \times_d \mathcal{B}$ is a uniform algebra.

Conversely, suppose that $\mathcal{A}$ is a uniform algebra. Let $\|\cdot\|$ be a complete uniform norm on $\mathcal{A}$. Define $|(a, b)| = \max\{\|a + b\|, \|b\|\}$ $((a, b) \in \mathcal{A} \times_d \mathcal{B})$. Then, by Lemma 2.5(ii), the norm $|\cdot|$ is a uniform norm on $\mathcal{A} \times_d \mathcal{B}$. We show that $|\cdot|$ is complete on $\mathcal{A} \times_d \mathcal{B}$. Let $((a_n, b_n))$ be a Cauchy sequence in $(\mathcal{A} \times_d \mathcal{B}, |\cdot|)$. Then

$$\|a_n\| \leq \|a_n + b_n\| + \|b_n\| \leq 2\max\{\|a_n + b_n\|, \|b_n\|\} = 2|(a_n, b_n)| \quad (n \in \mathbb{N}).$$

This implies that (a_n) is a Cauchy sequence in $(\mathcal{A}, \|\cdot\|)$. Since $\|\cdot\|$ is a complete norm on $\mathcal{A}$, the sequence (a_n) converges to some $a \in \mathcal{A}$. Similarly, the sequence (b_n) converges to some $b \in \mathcal{B}$. Hence, the sequence $((a_n, b_n))$ converges to (a, b) in $|\cdot|$. Thus $|\cdot|$ is a complete uniform norm on $\mathcal{A} \times_d \mathcal{B}$. □

3. Gelfand Space and UUNP

Throughout this section $\mathcal{A}$ is a semisimple commutative Banach algebra and $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$. In this section, we characterized the Gelfand space $\Delta(\mathcal{A} \times_d \mathcal{B})$, interms of ΔA and ΔB. We use the following notations.

Notations: Let $\varphi \in \Delta(\mathcal{A})$. Define $\varphi^+, \varphi_\diamond : \mathcal{A} \times_d \mathcal{B} \longrightarrow \mathbb{C}$ as $\varphi^+((a,b)) := \varphi(a) + \varphi(b)$ and $\varphi_\diamond((a,b)) := \varphi(b)$ $((a,b) \in \mathcal{A} \times_d \mathcal{B})$. Let $F \subset \Delta(\mathcal{A})$. Define $F^+ := \{\varphi^+ : \varphi \in F\}$ and $F_\diamond := \{\varphi_\diamond : \varphi \in F\}$.

Theorem 3.1. $\Delta(\mathcal{A} \times_d \mathcal{B}) = \Delta^+(\mathcal{A}) \biguplus \Delta_\diamond(\mathcal{B})$.

Proof. Let $\widetilde{\eta} \in \Delta(\mathcal{A} \times_d \mathcal{B})$. Define $\varphi(a) = \widetilde{\eta}((a, 0))$ $(a \in \mathcal{A})$ and $\psi(b) = \widetilde{\eta}((0, b))$ $(b \in \mathcal{B})$. Then φ and ψ are linear maps on $\mathcal{A}$ and $\mathcal{B}$, respectively such that $\widetilde{\eta}((a, b)) = \varphi(a) + \psi(b)$ $((a, b) \in \mathcal{A} \times_d \mathcal{B})$. Now, for $(a, b), (c, d) \in \mathcal{A} \times_d \mathcal{B}$,

$$
\begin{aligned}
& \widetilde{\eta}[(a,b)(c,d)] = \widetilde{\eta}((a,b))\widetilde{\eta}((c,d)) \\
\Rightarrow\ & \widetilde{\eta}((ac + ad + bc, bd)) = (\varphi(a) + \psi(b))(\varphi(c) + \psi(d)) \\
\Rightarrow\ & \varphi(ac + ad + bc) + \psi(bd) = \varphi(a)\varphi(c) + \varphi(a)\psi(d) \\
& \qquad + \psi(b)\varphi(a) + \psi(b)\psi(d). \qquad (3.1)
\end{aligned}
$$

Now, if $\varphi \equiv 0$ on $\mathcal{A}$, then ψ must be nonzero on $\mathcal{B}$ and from Equation (3.1), we get $\psi(bd) = \psi(b)\psi(d)$. Thus $\psi \in \Delta(\mathcal{B})$. In this case, $\widetilde{\eta}((a, b)) = \psi(b) = \psi_\diamond((a, b))$ $((a, b) \in \mathcal{A} \times_d \mathcal{B})$. Thus $\widetilde{\eta} = \psi_\diamond \in \Delta_\diamond(\mathcal{B})$. If $\varphi \neq 0$ on $\mathcal{A}$, then there exists $a \in \mathcal{A}$ such that $\varphi(a) \neq 0$. Now, taking $b = d = 0$ in Equation (3.1), we get $\varphi(ac) = \varphi(a)\varphi(c)$. Which means $\varphi \in \Delta(\mathcal{A})$. Using this in Equation (3.1), we get, $\varphi(a)\varphi(d) + \varphi(b)\varphi(c) = \varphi(a)\psi(d) + \varphi(c)\psi(b)$. Taking $a = c$ and $b = d$, we get $\varphi(b) = \psi(b)$. Hence $\varphi = \psi$ on $\mathcal{B}$. Therefore, $\widetilde{\eta}((a, b)) = \varphi(a) + \psi(b) = \varphi(a) + \varphi(b) = \varphi^+((a, b))$ $((a, b) \in \mathcal{A} \times \mathcal{B})$. Thus, in this case, $\widetilde{\eta} = \varphi^+ \in \Delta^+(\mathcal{A})$. Thus $\Delta(\mathcal{A} \times_d \mathcal{B}) \subset \Delta^+(\mathcal{A}) \biguplus \Delta_\diamond(\mathcal{B})$. The reverse inclusion is trivial.

Next, we show that $\Delta^+(\mathcal{A})$ and $\Delta_\diamond(\mathcal{B})$ are open in $(\Delta^+(\mathcal{A}) \uplus \Delta_\diamond(\mathcal{B}), \mathcal{T}_g)$. Let $\varphi^+ \in \Delta^+(\mathcal{A})$. Then there exists $a \in \mathcal{A}$ such that $\varphi(a) \neq 0$. Let $\epsilon = |\varphi(a)|/2$ and $\widetilde{U} = U(\varphi^+, \epsilon, (a, 0))$. Then

$$
\begin{aligned}
\widetilde{U} &= \{\widetilde{\eta} \in \Delta(\mathcal{A} \times_d \mathcal{B}) : |\widetilde{\eta}((a,0)) - \varphi^+((a,0))| < \epsilon\} \\
&= \{\widetilde{\eta} \in \Delta(\mathcal{A} \times_d \mathcal{B}) : |\widetilde{\eta}((a,0)) - \varphi(a)| < \epsilon\}.
\end{aligned}
$$

If $\psi_\diamond \in \widetilde{U}$ for some $\psi \in \Delta(\mathcal{B})$, then $|\varphi(a)| < \epsilon$. This is not possible. Hence, in this case $\widetilde{U} \subset \Delta^+(\mathcal{A})$. This shows that $\Delta^+(\mathcal{A})$ is open in $(\Delta^+(\mathcal{A}) \uplus \Delta_\diamond(\mathcal{B}), \mathcal{T}_g)$.

Now, if possible suppose $\psi_0 \in \Delta_\diamond(\mathcal{B})$ be in the closure of $\Delta^+(\mathcal{A})$. Then there exists a net $(\varphi^+_\alpha) \in \Delta^+(\mathcal{A})$ converging to ψ_0, that is

$$
\varphi_\alpha(a) + \varphi_\alpha(b) \longrightarrow \psi(b) \quad ((a, b) \in \mathcal{A} \times_d \mathcal{B}).
$$

In particular, taking $b = 0$, $\varphi_\alpha(a) \longrightarrow 0$ $(a \in \mathcal{A})$. This implies $\psi = 0$. This is a contradiction. Hence $\Delta^+(\mathcal{A})$ is closed in $(\Delta^+(\mathcal{A}) \uplus \Delta_\diamond(\mathcal{B}), \mathcal{T}_g)$. Therefore, $\Delta_\diamond(\mathcal{B})$ is open in $(\Delta^+(\mathcal{A}) \uplus \Delta_\diamond(\mathcal{B}), \mathcal{T}_g)$. Now it follows that a subset W of $\Delta(\mathcal{A} \times_d \mathcal{B})$ is open in the Gelfand topology if and only if it is open in the sum topology. $\square$

Theorem 3.2. [5, Corollary 3.3.4] *Let X be a locally compact Hausdorff space, and let $\mathcal{A}$ be a subalgebra of $C_0(X)$ which strongly separates the points of X. Then a point $x \in X$ belongs to the Shilov boundary of $\mathcal{A}$ if and only if given any open neighbourhood U of x, there exist $f \in \mathcal{A}$ such that $\|f|_{X\setminus U}\|_\infty < \|f|_U\|_\infty$.*

Theorem 3.3. *$\partial(\mathcal{A} \times_d \mathcal{B}) = \partial^+(\mathcal{A}) \biguplus \partial_\diamond(\mathcal{B})$.*

Proof. Let $\varphi_0 \in \partial\mathcal{A}$. Let $\widetilde{U}$ be a neighborhood of φ_0^+. Set $U = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{U}\}$. Then U is a neighborhood of φ_0. Therefore, there exists $a \in \mathcal{A}$ such that

$$\|\widehat{a}|_{\Delta(\mathcal{A})\setminus U}\|_\infty < \|\widehat{a}|_U\|_\infty.$$

If $\psi_\diamond \in \Delta(\mathcal{A} \times_d \mathcal{B}) \setminus \widetilde{U}$, then $(a, 0)^\wedge(\psi_\diamond) = 0$. If $\varphi^+ \in \Delta(\mathcal{A} \times_d \mathcal{B}) \setminus \widetilde{U}$, then $\varphi \in \Delta(\mathcal{A}) \setminus U$ and $|(a, 0)^\wedge(\varphi^+)| = |\varphi(a)|$. This gives $\|(a, 0)^\wedge|_{\Delta(\mathcal{A}\times_d\mathcal{B})\setminus\widetilde{U}}\|_\infty = \|\widehat{a}|_{\Delta(\mathcal{A})\setminus U}\|_\infty$. Also $(a, 0)^\wedge(\varphi^+) = \widehat{a}(\varphi)$ for every $\varphi^+ \in \widetilde{U}$. Hence,

$$\|(a, 0)^\wedge|_{\Delta(\mathcal{A}\times_d\mathcal{B})\setminus\widetilde{U}}\|_\infty = \|\widehat{a}\|_{\Delta(\mathcal{A})\setminus U} < \|\widehat{a}|_U\|_\infty = \|(a, 0)^\wedge|_{\widetilde{U}}\|_\infty.$$

Therefore, by Theorem 3.2, $\varphi_0^+ \in \partial(\mathcal{A} \times_d \mathcal{B})$. Thus $\partial^+(\mathcal{A}) \subset \partial(\mathcal{A} \times_d \mathcal{B})$.

Let $\psi_0 \in \partial\mathcal{B}$. Let $\widetilde{V}$ be a neighborhood of $(\psi_0)_\diamond$. Set $V = \{\psi \in \Delta(\mathcal{B}) : \psi_\diamond \in \widetilde{V}\}$. Then $\widetilde{V}$ is a neighborhood of ψ_0. Therefore there exists $b \in \mathcal{B}$ such that

$$\|\widehat{b}|_{\Delta(\mathcal{B})\setminus V}\|_\infty = \|\widehat{b}|_V\|_\infty.$$

If $\psi_\diamond \in \Delta(\mathcal{A} \times_d \mathcal{B}) \setminus \widetilde{V}$, then $(-b, b)^\wedge(\psi_\diamond) = \widehat{b}(\psi)$. If $\varphi^+ \in \Delta(\mathcal{A} \times_d \mathcal{B}) \setminus \widetilde{V}$, then $|(-b, b)^\wedge(\varphi^+)| = 0$. This gives $\|(-b, b)^\wedge|_{\Delta(\mathcal{A}\times_d\mathcal{B})\setminus\widetilde{V}}\|_\infty = \|\widehat{b}|_{\Delta(\mathcal{B})\setminus V}\|_\infty$. Also $(-b, b)^\wedge(\psi_\diamond) = \widehat{b}(\psi)$ for every $\psi_\diamond \in \widetilde{V}$. Hence,

$$\|(-b, b)^\wedge|_{\Delta(\mathcal{A}\times_d\mathcal{B})\setminus\widetilde{V}}\|_\infty < \|(-b, b)^\wedge|_{\widetilde{V}}\|_\infty.$$

Therefore, by Theorem 3.2, $\psi_{0\diamond} \in \partial(\mathcal{A} \times_d \mathcal{B})$. Thus $\partial_\diamond(\mathcal{B}) \subset \partial(\mathcal{A} \times_d \mathcal{B})$.

Let $\varphi_0^+ \in \partial(\mathcal{A} \times_d \mathcal{B})$. Let U be a neighborhood of $\varphi_0 \in \Delta(\mathcal{A})$. Then $\widetilde{U} = U^+$ is a neighborhood of φ_0^+ in $\Delta(\mathcal{A} \times_d \mathcal{B})$. Since $\varphi_0^+ \in \partial(\mathcal{A} \times_d \mathcal{B})$, there exists $(a, b) \in \mathcal{A} \times_d \mathcal{B}$ such that

$$\|(a, b)^\wedge|_{\Delta(\mathcal{A}\times_d\mathcal{B})\setminus\widetilde{U}}\|_\infty < \|(a, b)^\wedge|_{\widetilde{U}}\|_\infty.$$

This gives $\|(a + b)^\wedge|_{\Delta(\mathcal{A})\setminus U}\|_\infty < \|(a + b)^\wedge|_U\|_\infty$. Therefore $\varphi_0 \in \partial\mathcal{A}$.

Let $(\psi_0)_\diamond \in \partial(\mathcal{A} \times_d \mathcal{B})$. Let V be a neighborhood of $\psi_0 \in \Delta(\mathcal{B})$. Then $\widetilde{V} = V_\diamond$ is a neighborhood of $(\psi_0)_\diamond$ in $\Delta(\mathcal{A} \times_d \mathcal{B})$. Since $(\psi_0)_\diamond \in \partial(\mathcal{A} \times_d \mathcal{B})$, there exists $(a, b) \in \mathcal{A} \times_d \mathcal{B}$ such that $\|(a, b)^\wedge|_{\Delta(\mathcal{A}\times_d\mathcal{B})\setminus\widetilde{V}}\|_\infty < \|(a, b)^\wedge|_{\widetilde{V}}\|_\infty$. This gives $\|\widehat{b}|_{\Delta(\mathcal{B})\setminus V}\|_\infty < \|\widehat{b}|_V\|_\infty$. Therefore, by Theorem 3.2, $\psi_0 \in \partial\mathcal{B}$. Thus, from both cases, it follows that $\partial(\mathcal{A} \times_d \mathcal{B}) \subset \partial^+(\mathcal{A}) \biguplus \partial_\diamond(\mathcal{B})$. $\square$

4. Uniqueness and Separation Properties

Here we characterize UUNP and UC*NP of $\mathcal{A} \times_d \mathcal{B}$ in terms of $\mathcal{A}$ and $\mathcal{B}$. We start with the following two useful lemmas.

Lemma 4.1. *Let $\mathcal{A}$ be a semisimple, commutative Banach algebra and $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$. Let $\widetilde{F} \subset \Delta(\mathcal{A} \times_d \mathcal{B})$. Define $F_{\mathcal{A}} = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{F}\}$ and $F_{\mathcal{B}} = \{\varphi \in \Delta(\mathcal{B}) : \varphi_\diamond \in \widetilde{F}\}$. Then*

(1) $F_{\mathcal{A}}^+ \cup F_{\mathcal{B}\diamond} = \widetilde{F}$.

(2) *If $\widetilde{F}$ is closed, then $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ are closed in $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$, resp.*

(3) *If $\widetilde{F}$ is a set of uniqueness for $\mathcal{A} \times_d \mathcal{B}$, then $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ are sets of uniqueness for $\mathcal{A}$ and $\mathcal{B}$, respectively.*

Proof. (1). This is trivial.
(2). Let $\widetilde{F} \subset \Delta(\mathcal{A} \times_d \mathcal{B})$ be closed. Let $\varphi \in \overline{F_{\mathcal{A}}}$. Then there exists a net (φ_α) in $F_{\mathcal{A}}$ such that $\varphi_\alpha \longrightarrow \varphi$. Then $\varphi_\alpha^+ \longrightarrow \varphi^+$. Since $\varphi_\alpha^+ \in \widetilde{F}$ and $\widetilde{F}$ is closed, $\varphi^+ \in \widetilde{F}$. So that $\varphi \in F_{\mathcal{A}}$. Thus $F_{\mathcal{A}}$ is closed in $\Delta(\mathcal{A})$. Similarly, $F_{\mathcal{B}}$ is closed in $\Delta(\mathcal{B})$.
(3). Let $\widetilde{F}$ be a set of uniqueness for $\mathcal{A} \times_d \mathcal{B}$. Let $a \in \mathcal{A}$ such that $|\widehat{a}|_{F_{\mathcal{A}}} = 0$. Then $(a,0)^\wedge(\varphi^+) = \widehat{a}(\varphi) = 0$ $(\varphi \in F_{\mathcal{A}})$. Thus $(a,0)^\wedge = 0$ on $F_{\mathcal{A}}^+$. It is clear that $(a,0)^\wedge = 0$ on $F_{\mathcal{B}\diamond}$. Hence $(a,0)^\wedge = 0$ on $\widetilde{F}$. This implies $(a,0) = (0,0)$ as $\widetilde{F}$ is a set of uniqueness for $\mathcal{A} \times_d \mathcal{B}$. Thus $a = 0$. Hence, $F_{\mathcal{A}}$ is a set of uniqueness for $\mathcal{A}$.

Next, let $b \in \mathcal{B}$ such that $|\widehat{b}|_{F_{\mathcal{B}}} = 0$. Then $(-b,b)^\wedge(\psi_\diamond) = \widehat{b}(\psi) = 0$ $(\psi \in F_{\mathcal{B}})$. Therefore, $(-b,b)^\wedge|_{F_{\mathcal{B}\diamond}} = 0$. Also, $(-b,b)^\wedge(\varphi^+) = 0$ $(\varphi \in F_{\mathcal{A}})$. This means $(-b,b)^\wedge|_{F_{\mathcal{A}}^+} = 0$. Thus, $(-b,b)^\wedge|_{\widetilde{F}} = 0$. Since $\widetilde{F}$ is a set of uniqueness for $\mathcal{A} \times_d \mathcal{B}$, $(-b,b) = (0,0)$. i.e. $b = 0$. Thus $F_{\mathcal{B}}$ is a set of uniqueness for $\mathcal{B}$. □

Lemma 4.2. *Let $\mathcal{A}$ be a semisimple, commutative Banach algebra and let $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$.*

(1) *If F is a set of uniqueness for $\mathcal{A}$, then so is $F^+ \cup \Delta_\diamond(\mathcal{B})$ for $\mathcal{A} \times_d \mathcal{B}$;*

(2) *If G is a set of uniqueness for $\mathcal{B}$, then so is $\Delta^+(\mathcal{A}) \cup G_\diamond$ for $\mathcal{A} \times_d \mathcal{B}$.*

Proof. (1). Let $F \subset \Delta(\mathcal{A})$ be a set of uniqueness for $\mathcal{A}$. Let $(a,b) \in \mathcal{A} \times_d \mathcal{B}$ such that $|(a,b)^\wedge|_{F^+ \cup \Delta_\diamond(\mathcal{B})} = 0$. This implies $|(a,b)^\wedge|_{F^+} = |(a,b)^\wedge|_{\Delta_\diamond(\mathcal{B})} = 0$. In particular, $|\psi(b)| = |(a,b)^\wedge(\psi_\diamond)| = 0$ $(\psi \in \Delta(\mathcal{B}))$. Then semisimplicity of $\mathcal{B}$ implies $b = 0$. Therefore, $|\widehat{a}|_F = |(a,0)^\wedge|_{F^+} = |(a,b)^\wedge|_{F^+} = 0$. Hence, by the hypothesis, $a = 0$. Thus $F^+ \cup \Delta_\diamond(\mathcal{B})$ is a set of uniqueness for $\mathcal{A} \times_d \mathcal{B}$.
(2). This proof follows by similar arguments as in (1). □

Definition 4.3. [1, 2] An algebra $\mathcal{A}$ has

(1) *unique uniform norm property (UUNP)* if $\mathcal{A}$ has exactly one uniform norm.

(2) *unique C^*-norm property (UC*NP)* if $\mathcal{A}$ has exactly one C^* norm.

Theorem 4.4. *$\mathcal{A} \times_d \mathcal{B}$ has UUNP if and only if $\mathcal{A}$ and $\mathcal{B}$ have UUNP.*

Proof. Let $\mathcal{A} \times_d \mathcal{B}$ have UUNP. Let $F \subset \Delta(\mathcal{A})$ be a closed set of uniqueness for $\mathcal{A}$. Then, $F^+ \uplus \Delta_\diamond(\mathcal{B})$ is a closed subset of $\Delta^+(\mathcal{A}) \cup \Delta_\diamond(\mathcal{B})$. Moreover, by Lemma 4.2(i), it is also a set of uniqueness for

$\mathcal{A}\times_d\mathcal{B}$. Since $\mathcal{A}\times_d\mathcal{B}$ has UUNP, by [2, Theorem 2.3], we get $\partial^+(\mathcal{A})\uplus\partial_\diamond(\mathcal{B})\subset F^+\uplus\Delta_\diamond(\mathcal{B})$. This implies $\partial^+\mathcal{A}\subset F^+$ as $\Delta^+(\mathcal{A})$ and $\Delta_\diamond(\mathcal{B})$ are disjoint. Hence, we get $\partial\mathcal{A}\subset F$. Thus $\partial\mathcal{A}$ is the smallest closed set of uniqueness for $\mathcal{A}$. Therefore, by [2, Theorem 2.3], $\mathcal{A}$ has UUNP. Similarly, it follows that $\mathcal{B}$ has UUNP.

Conversely, suppose that $\mathcal{A}$ and $\mathcal{B}$ have UUNP. Let $\widetilde{F}\subset\Delta(\mathcal{A}\times_d\mathcal{B})$ be a closed set of uniqueness for $\mathcal{A}\times_d\mathcal{B}$. Then, by Lemma 4.1, $F_\mathcal{A}$ and $F_\mathcal{B}$ are closed sets of uniqueness for $\mathcal{A}$ and $\mathcal{B}$, respectively such that $F_\mathcal{A}^+\cup F_{\mathcal{B}\diamond}=\widetilde{F}$. Since $\mathcal{A}$ and $\mathcal{B}$ have UUNP, by [2, Theorem 2.3], $\partial\mathcal{A}\subset F_\mathcal{A}$ and $\partial\mathcal{B}\subset F_\mathcal{B}$. Therefore, $\partial^+(\mathcal{A})\subset F_\mathcal{A}^+$ and $\partial_\diamond(\mathcal{B})\subset F_{\mathcal{B}\diamond}$. Hence, $\partial(\mathcal{A}\times_d\mathcal{B})\subset\widetilde{F}$. Thus $\partial(\mathcal{A}\times_d\mathcal{B})$ is the smallest closed set of uniqueness for $\mathcal{A}\times_d\mathcal{B}$. Hence, again by [2, Theorem 2.3], $\mathcal{A}\times_d\mathcal{B}$ has UUNP. □

Theorem 4.5. *Let $\mathcal{A}$ be a $*$-semisimple, Banach $*$-algebra and let $\mathcal{B}$ be a closed $*$-subalgebra of $\mathcal{A}$.*

(1) *If $\mathcal{A}\times_d\mathcal{B}$ has UC*NP, then $\mathcal{A}$ and $\mathcal{B}$ have UC*NP;*

(2) *Suppose that $\mathcal{A}$ is commutative. If $\mathcal{A}$ and $\mathcal{B}$ have UC*NP, then $\mathcal{A}\times_d\mathcal{B}$ has UC*NP.*

Proof. (1). Let $\mathcal{A}\times_d\mathcal{B}$ have UC*NP. Let $|\cdot|_\mathcal{A}$ and $|\cdot|_\mathcal{B}$ be the largest C^*-norms on $\mathcal{A}$ and $\mathcal{B}$ respectively. Define $|(a,b)|=\max\{|a+b|_\mathcal{A},\ |b|_\mathcal{B}\}$ $((a,b)\in\mathcal{A}\times_d\mathcal{B})$. Then, by Theorem 2.5(iii), $|\cdot|$ is a C^*-norm on $\mathcal{A}\times_d\mathcal{B}$. Now, let $|||\cdot|||_\mathcal{A}$ be any C^*-norm on $\mathcal{A}$. Define $|||(a,b)|||=\max\{\,|||a+b|||_\mathcal{A},\ |b|_\mathcal{B}\,\}$ $((a,b)\in\mathcal{A}\times_d\mathcal{B})$. Then $|||\cdot|||$ is also a C^*-norm on $\mathcal{A}\times_d\mathcal{B}$. Hence, by the hypothesis, $|\cdot|=|||\cdot|||$ on $\mathcal{A}\times_d\mathcal{B}$. Now, $|||a|||_\mathcal{A}=|||(a,0)|||=|(a,0)|=|a|_\mathcal{A}$ $(a\in\mathcal{A})$. Thus $\mathcal{A}$ has UC*NP. By similar arguments, it follows that $\mathcal{B}$ has UC*NP.

(2). Suppose that $\mathcal{A}$ and $\mathcal{B}$ are commutative and both have UC*NP. Let $\widetilde{F}$ be a proper closed subset of $\Delta^{h+}(\mathcal{A})\uplus\Delta^h_\diamond(\mathcal{B})$. Then, by Lemma 4.1(ii), the corresponding sets $F_\mathcal{A}$ and $F_\mathcal{B}$ are closed subsets of $\Delta^h(\mathcal{A})$ and $\Delta^h(\mathcal{B})$, respectively and one of them has to be a proper subset. Suppose that $F_\mathcal{A}$ is a proper closed subset of $\Delta^h(\mathcal{A})$. Since $\mathcal{A}$ has UC*NP, by [1, Proposition 1.3], there exists a nonzero element $a\in\mathcal{A}$ such that $\widehat{a}|_{F_\mathcal{A}}=0$. Then $(a,0)^\wedge|_{\widetilde{F}}=\widehat{a}|_{F_\mathcal{A}}=0$. Similarly, if $F_\mathcal{B}$ is a proper closed subset of $\Delta^h(\mathcal{B})$, then there exists a nonzero element $b\in\mathcal{B}$ such that $\widehat{b}|_{F_\mathcal{B}}=0$. Then $(-b,b)^\wedge|_{\widetilde{F}}=\widehat{b}|_{F_\mathcal{B}}=0$. Thus in each case, we get a nonzero element in $\mathcal{A}\times_d\mathcal{B}$ whose Gelfand transform is zero on $\widetilde{F}$. Therefore, by [1, Proposition 1.3], $\mathcal{A}\times_d\mathcal{B}$ has UC*NP. □

Definition 4.6. $\mathcal{A}$ is *weakly regular* (WR) if for each proper closed set $F\subset\Delta(\mathcal{A})$, there exists $a\in\mathcal{A}$ such that $\widehat{a}|_F=0$.

Theorem 4.7. *$\mathcal{A}\times_d\mathcal{B}$ is WR if and only if $\mathcal{A}$ and $\mathcal{B}$ are WR.*

Proof. Let $\mathcal{A}\times_d\mathcal{B}$ be weakly regular. Let F be a proper closed subset of $\Delta(\mathcal{A})$. Then $F^+\uplus\Delta_\diamond(\mathcal{B})$ is a proper closed subset of $\Delta^+(\mathcal{A})\uplus\Delta_\diamond(\mathcal{B})$. Hence, by the hypothesis, there exists non-zero $(a,b)\in\mathcal{A}\times_d\mathcal{B}$ such that $(a,b)^\wedge|_{F^+\uplus\Delta_\diamond(\mathcal{B})}=0$. This implies $(a,b)^\wedge|_{F^+}=0$ and $(a,b)^\wedge|_{\Delta_\diamond(\mathcal{B})}=0$. Now, let $\psi\in\Delta(\mathcal{B})$. Then $\psi_\diamond\in\Delta_\diamond(\mathcal{B})$. Hence, $\psi(b)=\widehat{b}(\psi)=(a,b)^\wedge(\psi_\diamond)=0$. Since $\mathcal{B}$ is semisimple, $b=0$. Thus we must have $a\neq 0$. Also, $\widehat{a}|_F=(a,0)^\wedge|_{F^+}=(a,b)^\wedge|_{F^+}=0$. Thus $\mathcal{A}$ is weakly regular. Similarly, it can be proved that $\mathcal{B}$ is weakly regular.

Conversely, assume that $\mathcal{A}$ and $\mathcal{B}$ are weakly regular. Let $\widetilde{F}$ be a proper closed subset of $\Delta^+(\mathcal{A})\uplus\Delta_\diamond(\mathcal{B})$. Then, by Lemma 4.1, the corresponding sets $F_\mathcal{A}$ and $F_\mathcal{B}$ are closed in $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$, respectively such that $F^+_{\mathcal{A}} \cup F_{\mathcal{B}_\diamond} = \widetilde{F}$ and one of them must be proper. Suppose that $F_{\mathcal{A}}$ is a proper closed subset of $\Delta(\mathcal{A})$. Then, by the hypothesis, there exists non-zero $a \in \mathcal{A}$ such that $\widehat{a}|_{F_{\mathcal{A}}} = 0$. Therefore $(a,0)^\wedge|_{F^+_{\mathcal{A}}} = \widehat{a}|_{F_{\mathcal{A}}} = 0$. Also, it is obvious that $(a,0)^\wedge|_{F_{\mathcal{B}_\diamond}} = 0$. Thus $(a,0)^\wedge|_{\widetilde{F}} = 0$. Similarly, if $F_{\mathcal{B}}$ is a proper closed subset of $\Delta(\mathcal{B})$, then there exists a nonzero $b \in \mathcal{B}$ such that $(-b,b)^\wedge|_{\widetilde{F}} = 0$. Hence, $\mathcal{A} \times_d \mathcal{B}$ is weakly regular. □

Definition 4.8. $\mathcal{A}$ is *regular* if for every closed set $F \subset \Delta(\mathcal{A})$ and an element $\varphi \in \Delta(\mathcal{A}) \setminus F$, there exists an element $a \in \mathcal{A}$ such that $\widehat{a}(\varphi) = 1$ and $\widehat{a}|_F = 0$.

Theorem 4.9. *$\mathcal{A} \times_d \mathcal{B}$ is regular if and only if both $\mathcal{A}$ and $\mathcal{B}$ are regular.*

Proof. Let $\mathcal{A} \times_d \mathcal{B}$ be regular. Let F be a closed subset of $\Delta(\mathcal{A})$ and $\varphi \in \Delta(\mathcal{A}) \setminus F$. Then $F^+ \uplus \Delta_\diamond(\mathcal{B})$ is a closed subset of $\Delta^+(\mathcal{A}) \uplus \Delta_\diamond(\mathcal{B})$ and $\varphi^+ \notin F^+ \uplus \Delta_\diamond(\mathcal{B})$. Therefore, by the hypothesis, there exists $(a,b) \in \mathcal{A} \times_d \mathcal{B}$ such that $(a,b)^\wedge|_{F^+ \uplus \Delta_\diamond(\mathcal{B})} = 0$ and $(a,b)^\wedge(\varphi^+) = 1$. This gives $(a,b)^\wedge|_{F^+} = 0$ and $(a,b)^\wedge|_{\Delta_\diamond(\mathcal{B})} = 0$. Therefore, $\psi(b) = \widehat{b}(\psi) = (a,b)^\wedge(\psi_\diamond) = 0$ $(\psi \in \Delta(\mathcal{B}))$. Since $\mathcal{B}$ is semisimple, $b = 0$. Then, we must have $a \neq 0$. Also, $\widehat{a}|_F = (a,0)^\wedge|_{F^+} = (a,b)^\wedge|_{F^+} = 0$ and $\widehat{a}(\varphi) = (a,0)^\wedge(\varphi^+) = (a,b)^\wedge(\varphi^+) = 1$. Thus $\mathcal{A}$ is regular. By similar arguments it follows that $\mathcal{B}$ is regular.

Conversely, assume that $\mathcal{A}$ and $\mathcal{B}$ are regular. Let $\widetilde{F}$ be a closed subset of $\Delta^+(\mathcal{A}) \uplus \Delta_\diamond(\mathcal{B})$ and $\widetilde{\eta} \in (\Delta^+(\mathcal{A}) \uplus \Delta_\diamond(\mathcal{B})) \setminus \widetilde{F}$. Then, by Lemma 4.1, the corresponding sets $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ are closed in $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$, respectively such that $F^+_{\mathcal{A}} \cup F_{\mathcal{B}_\diamond} = \widetilde{F}$. Since $\widetilde{\eta} \notin \widetilde{F}$, either $\widetilde{\eta} \in \Delta^+(\mathcal{A}) \setminus F^+_{\mathcal{A}}$ or $\widetilde{\eta} \in \Delta_\diamond(\mathcal{B}) \setminus F_{\mathcal{B}_\diamond}$. Suppose $\widetilde{\eta} \in \Delta^+(\mathcal{A}) \setminus F^+_{\mathcal{A}}$. Then, $\widetilde{\eta} = \varphi^+$ for some $\varphi \in \Delta(\mathcal{A}) \setminus F_{\mathcal{A}}$. Since $\mathcal{A}$ is regular, there exists $a \in \mathcal{A}$ such that $\widehat{a}|_{F_{\mathcal{A}}} = 0$ and $\widehat{a}(\varphi) = 1$. Then $(a,0)^\wedge|_{\widetilde{F}} = (a,0)^\wedge|_{F^+_{\mathcal{A}} \cup F_{\mathcal{B}_\diamond}} = 0$ and $(a,0)^\wedge(\varphi^+) = \widehat{a}(\varphi) = 1$. Similarly if $\widetilde{\eta} \in \Delta_\diamond(\mathcal{B}) \setminus F_{\mathcal{B}_\diamond}$. Then $\widetilde{\eta} = \psi_\diamond$ for some $\psi \in \Delta(\mathcal{B})$. In this case, using regularity of $\mathcal{B}$, we get $b \in \mathcal{B}$ such that $(-b,b)^\wedge|_{\widetilde{F}} = 0$ and $(-b,b)^\wedge(\psi_\diamond) = \widehat{b}(\psi) = 1$. Hence $\mathcal{A} \times_d \mathcal{B}$ is regular. □

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