Mathematics Today Vol.34(A) (April 2018 - Special Issue) 195-201
ISSN 0976-3228, e-ISSN 2455-9601

Approximate solution of Nonlinear Second Order Diffusion Equation arising in Ground Water Infiltration Phenomenon using Power Series Method (PSM)

Pragneshkumar R. Makwana[1] and Amit K. Parikh[2]

[1]Research Scholar and Assistant professor, Department of Mathematics, C. U. Shah University, Wadhwan City, India.,
Email: m_ pragnesh@yahoo.co.in

[2]Professor and Principal, Department of Mathematics, Mehsana Urban Institute of Sciences, Ganpat University, Kherva, India.
Email: amit.parikh.maths@gmail.com

Abstract

The nonlinear second order diffusion equation arising in groundwater infiltration phenomenon which has been solved with appropriate conditions and height of water mound is obtained in the form of infinite series by using Power Series Method (PSM). The solution is obtained for height of water mound in unsaturated porous media with respect to distance and time. The numerical as well as graphical representation of the solution is given by MAT LAB coding. It is concluded that the solution is consistent with physical phenomenon.

Keywords: Nonlinear diffusion equation, Groundwater, Infiltration, Power Series Method
AMS Subject Classification (2010): 76S05; 40G10

1 Introduction

Groundwater is water that exists in the pore spaces and fractures in rocks and sediments beneath the Earth's surface. It originates as rainfall or snow, and then moves through the soil and rock into the groundwater system, where it eventually makes its way back to the surface streams, lakes, or oceans. Groundwater makes up about 1 percentage of the water on the Earths (most water is in Oceans). But, groundwater makes up to 35 times the amount of water in lakes and streams. Groundwater occurs everywhere beneath the Earth's surface, but it usually restricted to depth less than about 750 meters. The volume of groundwater is equivalent to a 55-meter thick layer spread out over the entire surface of the Earth.

Groundwater is extremely important to our way of life. Most drinking water supplies and often irrigation water for agricultural needs are drawn from underground sources. More than 90 percent of the liquid fresh water available on or near the earth's surface is groundwater. Hot groundwater can also be a source of energy. Groundwater is derived from rain and melting snow that percolate downward from the surface; it collects in the open pore spaces between soil particles or in cracks and fissures in bedrock. The process of percolation is called **infiltration**. The infiltration model was first developed by **J. Boussinesq(1903)** and is related to the original motivation of **(H. Darcy (1856)), (Polubarinova Kochina (1962)), (A. Scheidegger (1968)), (M. Muskat (1946)) and (J. Bear (1972))**. Different researchers have discussed this problem with a different point of views. **(A. Verma (1967))** discussed infiltration of incompressible fluid for inclined plain in heterogeneous porous media. **(M. Mehta (1977))** obtained the solution of singular perturbation technique of one dimensional flow in unsaturated porous media. **(M. Mehta and S. Yadav (2008))** discussed the solution of problem arising during vertical ground water recharge by spreading in slightly saturated porous media. **(M. Mehta and S. Meher (2010))** discussed the Adomian decomposition method for moisture content in one dimensional fluid flow through unsaturated porous media. **(M. Joshi and M. Mehta (2010))** apply the Group theoretic approach to the problem of one dimensional fluid flow through unsaturated porous method. **(A. Parikh and M. Mehta (2011))** discussed the atmospheric pressure in dry region and velocity of infiltered water in ground water infilteration phenomenon [2]. **(R. Borana and V. Pradhan (2013))** obtained the numerical solution of boussinesq equation arising in one-dimensional infiltration phenomenon by using finite difference method [15]. **(K. Patel and M. Mehta (2014))** obtained a solution of boussinesq's equation for infiltration phenomenon in unsaturated porous media by homotopy analysis method [8]. **(S. Pathak and T. Singh (2015))** obtained an analytic solution of mathematical model of boussinq's equation in homogeneous porous media during infiltration of groundwater flow [16]. **(N. Desai (2017))** obtained similarity solution of non-linear boussinesq's equation arising in infiltration of incompressible fluid flow [11].

In this paper, we have studied groundwater infiltration phenomenon in unsaturated porous media. The governing equation is nonlinear second order diffusion equation which has been solved by Power Series Method. Our purpose is to find height of the free surface of the water mound in soil at any distance X at any time $T > 0$.

2 Mathematical Formulation

To develop the mathematical formulation of the groundwater infiltration phenomenon consider the following assumptions.

(i) The stratum has height $h_{\max}$ and lies on top of a horizontal impervious bed, which we label as $z = 0$.

(ii) Ignore the transversal variable y and

(iii) The water mass which infiltrates the soil occupies a region described as $\Omega = \{(x, z) \in R : z \leq h(x, t)\}$.

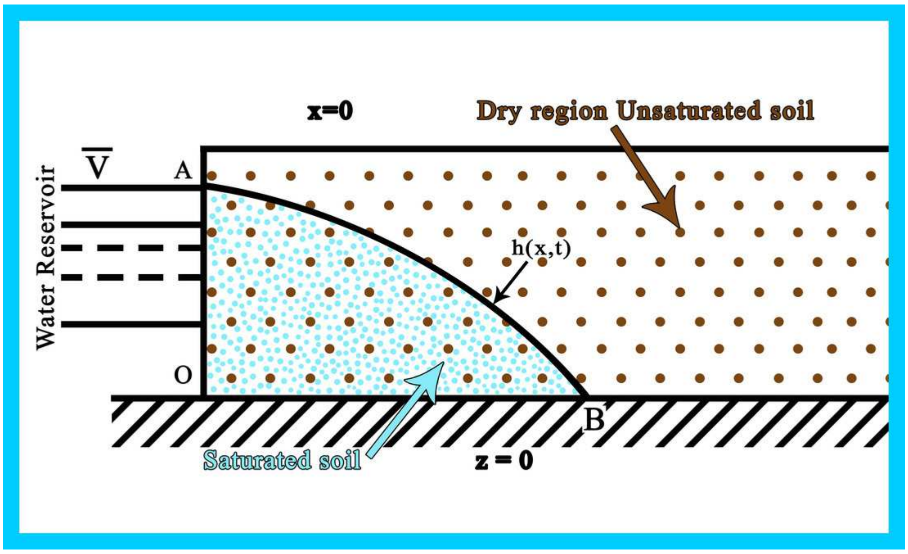

Figure 1: A Scheme of Groundwater Infiltration

In practical terms, we assume that there is no partial saturation. Clearly, $0 \leq h(x,t) \leq h_{\max}$ where, $h_{\max}$ is the maximum height of free surface and the free boundary function $h(x,t)$ is also an unknown. For the sake of simplicity and for the practical computation after introducing a suitable assumption, the hypothesis of almost horizontal flow, i.e., we assume that the flow has an almost horizontal speed. Here $u \approx (u, 0)$, so that h has small gradients. It follows that in the vertical component of the momentum equations

$$\rho\left(\frac{\partial u_z}{\partial t} + u \cdot \nabla u_z\right) = -\frac{\partial p}{\partial z} - \rho g \tag{1}$$

Neglecting the inertial term (the left-hand side) of equation (1), and integrating with respect to z, we obtain first approximation $p + \rho g z =$ constant. Now calculate the constant on the free surface $z = h(x,t)$. If we impose continuity of the pressure across the interface, we have $p = 0$ (assuming constant atmospheric pressure in the air that fills the pores of the dry region $z > h(x,t)$), we get

$$p = \rho g(h - z) \tag{2}$$

In other words, the pressure is determined by means of the hydrostatic approximation. Consider the mass conservation law for a section $S = (x, x+a) \times (0, C)$, we get

$$\phi \frac{\partial}{\partial t} \int_{x}^{x+a} \int_{0}^{h} dydx = -\int_{\partial s} u \cdot n \; dl \tag{3}$$

where ϕ the porosity of the medium, i.e., the fraction of volume available for the flow circulation, and u is the velocity, which obeys Darcys law in the form that includes gravity effects

$$u = -\frac{k}{\mu} \nabla(p + \rho g z) \tag{4}$$

On the right-hand lateral surface we have $u \cdot n \approx (u, 0) \cdot (1, 0) = u$ i.e. $-\left(\frac{k}{\mu}\right) \cdot \frac{\partial p}{\partial x}$ while on the left-hand side we have $-u$. Using the formula for p and differentiating with respect to x, we get

$$a\phi \frac{\partial h}{\partial t} = \frac{\rho g k}{\mu} \frac{\partial}{\partial x} \int_{0}^{h} \frac{\partial}{\partial x} h \; dz \tag{5}$$

Thus, we obtained non-linear second order diffusion equation (Boussinesq's equation) as

$$\frac{\partial h}{\partial t} = \frac{\rho g k}{a\mu\phi} \frac{\partial^2}{\partial x^2} h^2 \tag{6}$$

with an initial and boundary condition as

$$h(x,0) = h(x), \qquad t = 0, x > 0 \tag{7}$$

$$h(0,t) = h_{\max}, \qquad x = 0, t > 0 \tag{8}$$

Choosing new dimensionless variables as

$$X = \frac{x}{L} \ \text{ and } \ T = \frac{2\rho g k}{L^2 a\mu\phi} t \tag{9}$$

Then, equation (6) is in the following form

$$\frac{\partial h}{\partial T} = h \frac{\partial^2 h}{\partial X^2} + \left(\frac{\partial h}{\partial X}\right)^2, \tag{10}$$

with initial and boundary conditions as

$$h(X,0) = e^{-X}, \qquad T = 0, X > 0 \tag{11}$$

$$h(0,T) = h_{\max}, \qquad X = 0, T > 0 \tag{12}$$

Since condition (11) is sufficient to solve problem by Power Series Method. Hence we discard condition (12).

3 Solution by Power Series Method (PSM)[14]

Let us assume the solution of (10) as a power series in X and T as given below

$$h(X,T) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij} X^i T^j \tag{13}$$

By partial differentiating both side of (13) with respect to X and T, we get series expansion of $\frac{\partial h}{\partial X}$ and $\frac{\partial h}{\partial T}$ as follows:

$$\frac{\partial h}{\partial X} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (i+1)\, c_{(i+1)j} X^i T^j \tag{14}$$

$$\frac{\partial h}{\partial T} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (j+1)\, c_{i(j+1)} X^i T^j \tag{15}$$

Also the series expansion of $\left(\frac{\partial h}{\partial X}\right)^2$, $\frac{\partial^2 h}{\partial X^2}$ and $h \frac{\partial^2 h}{\partial X^2}$ are as follows:

$$\left(\frac{\partial h}{\partial X}\right)^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left[\sum_{s=0}^{j}\sum_{t=0}^{i} (t+1)(i-t+1)\, c_{(t+1)s}\, c_{(i-t+1)(j-s)}\right] X^i T^j \tag{16}$$

$$\frac{\partial^2 h}{\partial X^2} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (i+1)(i+2)\, c_{(i+2)j} X^i T^j \tag{17}$$

$$h \frac{\partial^2 h}{\partial X^2} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left[\sum_{s=0}^{j}\sum_{t=0}^{i} (i-t+1)(i-t+2)\, c_{ts}\, c_{(i-t+2)(j-s)}\right] X^i T^j \tag{18}$$

Substituting equations (15), (16) and (18) into (10), to obtained the recurrence relation:

$$c_{i(j+1)} = \frac{1}{(j+1)} \left[\sum_{s=0}^{j} \sum_{t=0}^{i} \left((i-t+1)(i-t+2)\ c_{ts}\ c_{(i-t+2)(j-s)} + (t+1)(i-t+1)\ c_{(t+1)s}\ c_{(i-t+1)(j-s)} \right) \right],\ \forall\ i,j \geq 0 \tag{19}$$

where,

$$c_{i0} = \frac{(-1)^i}{i!},\ \forall\ i \geq 0. \tag{20}$$

By applying the recurrence relation (19) for several values of i and j the approximate series solution obtained is as follows,

$$h(X,T) = 1 - X + \frac{1}{2}X^2 - \frac{1}{6}X^3 + 2T - 4XT + 4X^2T - \frac{8}{3}X^3T + \dots \tag{21}$$

4 Numerical and Graphical Representation

Numerical and graphical presentations of a equation (21) have been obtained by MAT LAB. The following Table (1) represents the numerical values for height of free surface for different distance X for fixed time $T = 0.1, 0.2, 0.3, 0.4, 0.5$. Figure (2) shows the graph of Hight h vs. Distance X for fixed time $T = 0.1, 0.2, 0.3, 0.4, 0.5$.

$h(X,T)$					
X	$T = 0.1$	$T = 0.2$	$T = 0.3$	$T = 0.4$	$T = 0.5$
0.0	1.2000	1.4000	1.6000	1.8000	2.0000
0.1	1.0686	1.2323	1.3960	1.5598	1.7235
0.2	0.9525	1.0864	1.2203	1.3541	1.4880
0.3	0.8493	0.9581	1.0669	1.1757	1.2845
0.4	0.7563	0.8432	0.9301	1.0171	1.1040
0.5	0.6708	0.7375	0.8042	0.8708	0.9375
0.6	0.5904	0.6368	0.6832	0.7296	0.7760
0.7	0.5124	0.5369	0.5614	0.5860	0.6105
0.8	0.4341	0.4336	0.4331	0.4325	0.4320

Table 1: Numerical values for $h(X,T)$ corresponding to X and T

5 Conclusion

The equation (21) represents height $h(X,T)$ of free surface in unsaturated porous media for any distance X and time $T > 0$. The solution is obtained in the form of infinite series by using Power Series Method. Figure (2), here we can conclude that the height $h(X,T)$ decreases when distance X increases for fixed time level $T = 0.1, 0.2, 0.3, 0.4, 0.5$. Which shows that the solution is consistent with physical nature of the phenomenon.

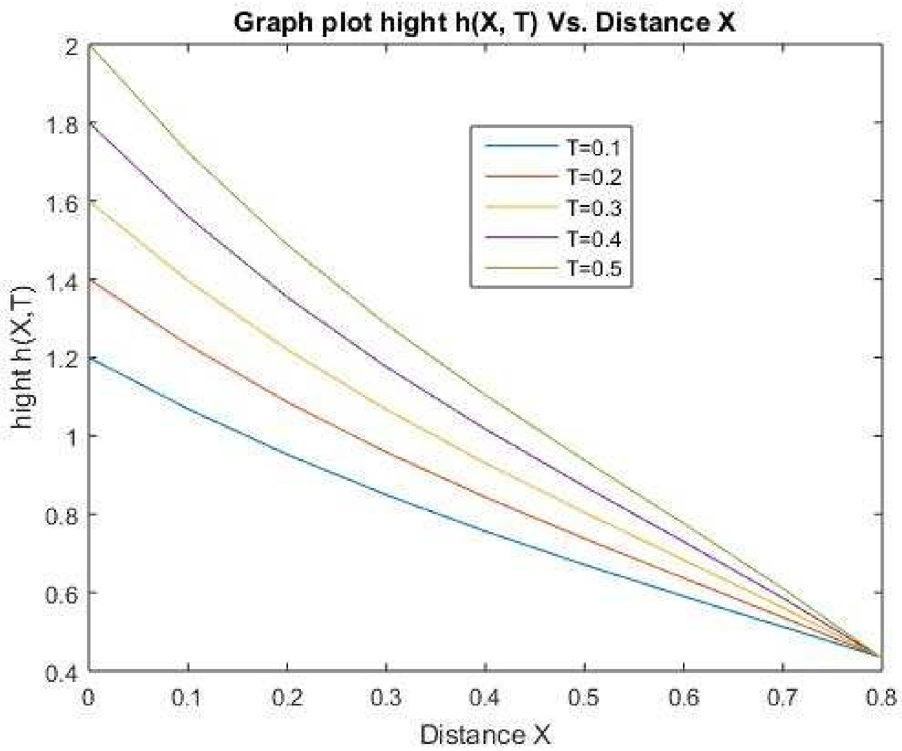

Figure 2: Hight $h(X, T)$ at different distance X with a fix time level $T = 0.1, 0.2, 0.3, 0.4, 0.5$

References

[1] Scheidegger, A. E. (1960), *The Physics of Flow through Porous Media*, University of Toronto Press.

[2] Parikh, A. K., Mehta, M. N. (2011), *The atmospheric pressure in dry region and velocity of infiltered water in ground water infilteration phenomenon*, International Journal of Applied Mathematics and Mechanics, 7 (13), 77-90.

[3] Nuseir, A. S., Al-Hasoon, A. (2012), *Power Series Solutions for Nonlinear Systems of Partial Differential Equations*, Applied Mathematical Sciences, 6 (104), 5147-5159.

[4] Lpez-Sandoval, E., Mello, A. (2016), *Power Series Solution to Non-linear Partial Differential Equations: Accuracy and Simplicity*, Transactions on Mathematics, 2 (2), 07-35.

[5] Darcy, H. (1856), *Les Fontaines Publiques de la Ville de Dijon*, Delmont, Paris. 305-401.

[6] Boussinesq, J. (1903-04), *Recherches thoriques sur lcoulement des nappes deau infiltres dans le sol et sur le dbit des sources*, Journal. Math. Pures Appl., 10, 5-78.

[7] Bear, J. (1972), *Dynamics of Fluids in Porous Media*, American Elsevier Publishing Company, Inc.

[8] Patel, K. K., Mehta, M. N. (2014), *A Solution of Boussinesq's equation for infiltration phenomenon in unsaturated porous media by Homotopy Analysis Method*, International organization of Scientific Research, 4 (2), 1-8.

[9] Muskat, M. (1973), *The Flow of Homogeneous Fluids Through Porous Media*, McGraw-Hill, New York.

[10] Mehta, M. N., Meher, S. K. (2010), *Adomian decomposition method for moisture content in one dimensional fluid flow through unsaturated porous media*, International Journal of Applied Mathematics and Mechanics, 6 (7), 13-23.

[11] Desai, N. B. (2017), *Similarity solution of non-linear boussinesq's equation arising in infiltration of incompressible fluid flow*, International Journals of Advanced Research in Computer Science and Software Engineering, 7 (6), 59-67.

[12] Roopashree, N. S., Nargund, A. L. (2016), *Power Series Solution of Non-linear Partial Differential Equations*, International Journal of Mathematics and Computer Research, 4 (6), 1514-1520.

[13] Polubarinova-Kochina, P. Ya. (1962), *Theory of Groundwater Movement*, Princeton University Press, Princeton.

[14] Makwana, P. R., Parikh, A. K. (2017), *Approximate Solution of Nonlinear Diffusion Equation Using Power Series Method (PSM)*, International Journal of Current Advanced Research, 06 (10), 6374-6377.

[15] Borana, R. N., Pradhan, V. H. (2013), *Numerical solution of boussinesq equation arising in one-dimensional infiltration phenomenon by using finite difference method*, International Journal of Research in Engineering and Technology, 2 (8), 202-209.

[16] Pathak, S. P., and Singh, T. (2015), *An Analytic Solution of Mathematical Model of Boussinqs Equation in Homogeneous Porous Media During Infiltration of Groundwater Flow*, Journal of Geography, Environment and Earth Science International, 3 (2), 1-8.