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On the operators defined by Lupaş with some parameters based on q-integers

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Abstract

The aim of this paper, introduce q-analogue of a sequence of linear and positive operators with two parameters which was introduced by A. Lupaş in 1995. First, we estimate moments of the operators and then prove a basic convergence theorem. Next, a local approximation theorem is established. Further, we study the rate of convergence and weighted approximation theorem for these operators.

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1 Introduction

The space of real valued continuous functions on the interval $\mathbb{R}^+ = [0,\infty)$ is denoted by $C(\mathbb{R}^+)$. The space of real valued bounded continuous functions on the interval $\mathbb{R}^+$ is denoted by $C_B(\mathbb{R}^+)$ with the norm $\|f\|_\infty = \sup\limits_{x\in\mathbb{R}^+} |f(x)|$ is a Banach space.
Consider the weight function $\rho_\lambda : \mathbb{R}^+ \to [1,\infty)$, $\rho_\lambda(x) = 1 + x^{2+\lambda} (\lambda > 0)$, we define the space

$$C_{\rho_\lambda}(\mathbb{R}^+) = \left\{ f \in C(\mathbb{R}^+) : \frac{f(x)}{\rho_\lambda(x)} \text{ is convergent as } x \to \infty \right\}$$

endowed with the usual norm $\|\cdot\|_{\rho_\lambda}$, $\|f\|_{\rho_\lambda} = \sup\limits_{x\in\mathbb{R}^+} \dfrac{|f(x)|}{\rho_\lambda(x)}$.
In 2016, Singh *et* al. [1] introduced the following sequence of positive linear operators as: For $f : \mathbb{R}^+ \to \mathbb{R}$,

$$L_{n,q}(f,x) = 2^{-[n]_q x} \sum_{k=0}^{\infty} \frac{([n]_q\, x)_k}{2^k [k]_q!} f\left(\frac{[k]_q}{[n]_q}\right),\ x \geq 0, \tag{1.1}$$

where $(\lambda)_0 = 1,\ (\lambda)_k = \lambda(\lambda+1)\ldots(\lambda+k-1),\ k \geq 1$. Before proceeding further, let us give some basic definitions and notations from q-calculus. Details on q-integers can be found in [2, 3].
Let $q > 0$, for each nonnegative integer k, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are defined as

$$[k]_q := \begin{cases} \dfrac{1-q^k}{1-q}, & q \neq 1 \\ k, & q = 1 \end{cases}$$

and

$$[k]_q! := \begin{cases} [k]_q[k-1]_q \dots [1]_q & k \geq 1 \\ 1, & k = 0 \end{cases},$$

respectively. For $q > 0$ and integers $n,\ k,\ n \geq k \geq 0$, we have

$[k+1]_q = 1 + q[k]_q$ and $[k]_q + q^k[n-k]_q = [n]_q$.

One should observe that, for $q = 1$, the operators (1.1) reduce to the operators introduced by Lupaş [4] as follows:

$$L_n(f,x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right),\ x \geq 0. \tag{1.2}$$

Some approximation properties of the operators (1.2) was studied in [5]. The Durrmeyer variant of the operators (1.2) was studied very recently in [6]. In [7] Agratini modified the operators (1.2) into integral form in Kantorovich sense and established their approximation properties. Recently, statistical approximation processes and some direct results of the operators L_n have been studied in [8].The Jain type variant of the operators (1.2) was established by Patel and Mishra [9].

In this manuscript, we modified the operators (1.1) in two parameters α and β with $0 \leq \alpha \leq \beta$. Similar type of generalization of positive linear operators is known as Stancu type generalization. These type of generalization can be found in [10, 11, 12, 13, 14, 15, 16, 17] for various other operators. Motivated by this, we modified the operators (1.2) as follows:

For $0 \leq \alpha \leq \beta,\ f \in C_{\rho_0}(\mathbb{R}^+),\ x \in \mathbb{R}^+$

$$L_{n,q}^{\alpha,\beta}(f,x) = 2^{-[n]_q x} \sum_{k=0}^{\infty} \frac{([n]_q\, x)_k}{2^k [k]_q!} f\left(\frac{[k]_q + \alpha}{[n]_q + \beta}\right). \tag{1.3}$$

Lemma 1.1 ([1]) *The following relations hold:*

$$L_{n,q}(1,x) = 1; L_{n,q}(t,x) = x; L_{n,q}\left(t^2,x\right) = qx^2 + \frac{1+q}{[n]_q}x.$$

Lemma 1.2 *The following relations hold:*

$$L_{n,q}^{\alpha,\beta}(1,x) = x;\ L_{n,q}^{\alpha,\beta}(t,x) = \frac{[n]_q\, x + \alpha}{[n]_q + \beta};$$

$$L_{n,q}^{\alpha,\beta}\left(t^2,x\right) = \frac{q[n]_q^2 x^2 + (2\alpha + 1 + q)[n]_q x + \alpha^2}{\left([n]_q + \beta\right)^2}.$$

Proof. We have

$$L_{n,q}^{\alpha,\beta}(1,x) = L_{n,q}(1,x) = 1;$$

$$\begin{aligned}
L_{n,q}^{\alpha,\beta}(t,x) &= 2^{-[n]_q x}\sum_{k=0}^{\infty}\frac{([n]_q\,x)_k}{2^k[k]_q!}\left(\frac{[k]_q+\alpha}{[n]_q+\beta}\right)\\
&= \frac{2^{-[n]_q\,x}}{[n]_q+\beta}\sum_{k=1}^{\infty}\frac{([n]_q\,x)_k}{2^k[k-1]_q!}+\frac{\alpha}{[n]_q+\beta}\\
&= \frac{2^{-[n]_q\,x-1}}{[n]_q+\beta}\sum_{k=1}^{\infty}\frac{[n]_q x\,([n]_q\,x+1)_{k-1}}{2^{k-1}[k-1]_q!}+\frac{\alpha}{[n]_q+\beta}\\
&= \frac{[n]_q x\,2^{-([n]_q\,x+1)}}{[n]_q+\beta}\sum_{k=0}^{\infty}\frac{([n]_q\,x+1)_k}{2^k[k]_q!}+\frac{\alpha}{[n]_q+\beta}\\
&= \frac{[n]_q\,x+\alpha}{[n]_q+\beta}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
L_{n,q}^{\alpha,\beta}\left(t^2,x\right) &= 2^{-[n]_q x}\sum_{k=0}^{\infty}\frac{([n]_q x)_k}{2^k[k]_q!}\frac{[k]_q^2+2[k]_q\alpha+\alpha^2}{\left([n]_q+\beta\right)^2}\\
&= \frac{2^{-[n]_q x}}{\left([n]_q+\beta\right)^2}\sum_{k=0}^{\infty}\frac{[n]_q x\,([n]_q\,x+1)_{k-1}}{2^k[k]_q[k-1]_q!}[k]_q^2\\
&\quad+\frac{2^{-[n]_q x}}{\left([n]_q+\beta\right)^2}\sum_{k=0}^{\infty}\frac{[n]_q x\,([n]_q\,x+1)_{k-1}}{2^k[k]_q[k-1]_q!}2[k]_q\alpha+\frac{\alpha^2}{\left([n]_q+\beta\right)^2}\\
&= \frac{[n]_q x 2^{-[n]_q x-1}}{\left([n]_q+\beta\right)^2}\sum_{k=0}^{\infty}\frac{([n]_q\,x+1)_k}{2^k[k]_q!}[k+1]_q\\
&\quad+\frac{2\alpha\,[n]_q x 2^{-([n]_q x+1)}}{\left([n]_q+\beta\right)^2}\sum_{k=0}^{\infty}\frac{([n]_q\,x+1)_k}{2^k[k]_q!}+\frac{\alpha^2}{\left([n]_q+\beta\right)^2}\\
&= \frac{[n]_q x 2^{-[n]_q x-1}}{\left([n]_q+\beta\right)^2}\sum_{k=0}^{\infty}\frac{([n]_q\,x+1)_k(1+q\,[k]_q)}{2^k[k]_q!}+\frac{2\alpha\,[n]_q x+\alpha^2}{\left([n]_q+\beta\right)^2}\\
&= \frac{q[n]_q x 2^{-[n]_q x-1}}{\left([n]_q+\beta\right)^2}\sum_{k=0}^{\infty}\frac{([n]_q\,x+1)_k[k]_q}{2^k[k]_q!}+\frac{(2\alpha+1)\,[n]_q x+\alpha^2}{\left([n]_q+\beta\right)^2}\\
&= \frac{q[n]_q x 2^{-[n]_q x-2}}{\left([n]_q+\beta\right)^2}\sum_{k=1}^{\infty}\frac{([n]_q\,x+1)([n]_q\,x+2)_{k-1}}{2^{k-1}[k-1]_q!}+\frac{(2\alpha+1)\,[n]_q x+\alpha^2}{\left([n]_q+\beta\right)^2}\\
&= \frac{q[n]_q^2x^2+(2\alpha+1+q)\,[n]_q x+\alpha^2}{\left([n]_q+\beta\right)^2},
\end{aligned}$$

which completes the proof of the Lemma 1.2.

Definition 1. Let $0<q<1$, using linear properties of the operators $L_{n,q}^{\alpha,\beta}$, for $x\in\mathbb{R}^+$, we have

$$\tau_{n,q}^{1,\alpha,\beta}(x)=L_{n,q}^{\alpha,\beta}(t-x,x)=\frac{\alpha-\beta x}{[n]_q+\beta};$$

$$\tau_{n,q}^{2,\alpha,\beta}(x)=L_{n,q}^{\alpha,\beta}\left((t-x)^2,x\right)=\frac{((q-1)\,[n]_q^2+\beta^2)x^2+\left(\,[2]_q\,[n]_q-2\alpha\beta\right)x+\alpha^2}{\left([n]_q+\beta\right)^2}.$$

2 Pointwise Convergence

Theorem 2.1 *Let $f \in C_{\rho_0}(\mathbb{R}^+)$ and q_n be a real sequence in $(0,1)$ such that $q_n \to 1$ and $q_n^n \to 0$ as $n \to \infty$. Then, for any compact set $K \subset \mathbb{R}^+$, we have*

$$\lim_{n\to\infty} L_{n,q_n}^{\alpha,\beta}(f,x) = f(x) \text{ uniformly in } x \in K.$$

Proof: The proof is based on the well-known Korovkin theorem regarding the convergence of a sequence of linear positive operators. So, it is enough to prove the conditions

$$\lim_{n\to\infty} L_{n,q_n}^{\alpha,\beta}(t^m,x) = x^m, \quad m = 0,1,2.$$

Now, using Lemma 1.2, we obtain

$$\lim_{n\to\infty} L_{n,q_n}^{\alpha,\beta}(1,x) = 1.$$

Also,

$$\lim_{n\to\infty} L_{n,q_n}^{\alpha,\beta}(t,x) = \lim_{n\to\infty}\left(\frac{[n]_{q_n}x}{[n]_{q_n}+\beta} + \frac{\alpha}{[n]_{q_n}+\beta}\right) = x.$$

Similarly,

$$\lim_{n\to\infty} L_{n,q_n}^{\alpha,\beta}\left(t^2,x\right) = \lim_{n\to\infty}\left(\frac{q_n[n]_{q_n}^2 x^2 + \left(2\alpha+[2]_{q_n}\right)[n]_{q_n} x + \alpha^2}{\left([n]_{q_n}+\beta\right)^2}\right) = x^2.$$

This completes the proof of Theorem 2.1.

3 Local Approximation

The Peetre's K-functional is defined by

$$K_2(f\ ,\ \delta) = \inf_{g\in C_B^2(\mathbb{R}^+)}\{\|f\ -\ g\|_\infty\ + \delta\|g''\|_\infty\},$$

where $C_B^2(\mathbb{R}) = \left\{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\right\}$. By [17], there exists a positive constant $C > 0$ such that $K_2(f\ ,\ \delta) \le C\omega_2\left(f\ ,\ \sqrt{\delta}\right)$, $\delta > 0$, where the second-order modulus of smoothness is given by

$$\omega_2\left(f\ ,\ \sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{0\le x<\infty} |f(x+2h) - 2f(x+h) + f(x)|.$$

Also, for $f \in C_B(\mathbb{R}^+)$ the usual modulus of continuity is given by

$$\omega(f,\ \delta) = \sup_{0<h\le\delta}\ \sup_{0\le x<\infty} |f(x+h) - f(x)|.$$

Theorem 3.1 *Suppose that $f \in C_B(\mathbb{R}^+)$ and $0 < q < 1$. Then for all $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$, there exists an absolute constant $C > 0$ such that*

$$|L_{n,q}^{\alpha,\beta}(f,x) - f(x)| \le C\omega_2\left(f, \sqrt{\tau_{n,q}^{2,\alpha,\beta}(x) + \left(\tau_{n,q}^{1,\alpha,\beta}(x)\right)^2}\right) + \omega\left(f, |\tau_{n,q}^{1,\alpha,\beta}(x)|\right)$$

Proof: We are introducing the auxiliary operators as follows

$$\tilde{L}_{n,q}^{\alpha,\beta}(f,x) = L_{n,q}^{\alpha,\beta}(f,x) - f\left(\frac{[n]_q\ x+\alpha}{[n]_q+\beta}\right) + f(x).$$

Suppose that $g \in C_B^2(\mathbb{R}^+)$ and $x, t \in \mathbb{R}^+$. From Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u) g''(u)\, du.$$

Applying $\tilde{L}_{n,q}^{\alpha,\beta}$, we get

$$\tilde{L}_{n,q}^{\alpha,\beta}(g,x) - g(x) = g'(x)\tilde{L}_{n,q}^{\alpha,\beta}(t-x,x) + \tilde{L}_{n,q}^{\alpha,\beta}\left(\int_x^t (t-u) g''(u)\, du, x\right).$$

Using Lemma 1.2, we obtain

$$\begin{aligned}
\left|\tilde{L}_{n,q}^{\alpha,\beta}(g,x) - g(x)\right| &\leq \tilde{L}_{n,q}^{\alpha,\beta}\left(\left|\int_x^t (t-u) g''(u)\, du\right|, x\right)\\
&\leq L_{n,q}^{\alpha,\beta}\left((t-x)^2, x\right)\|g''\|_\infty \quad + \left|\int_x^{\frac{[n]_q\ x+\alpha}{[n]_q+\beta}} \left(\frac{[n]_q\ x+\alpha}{[n]_q+\beta} - u\right) g''(u)\, du\right|\\
&\leq \left[\tau_{n,q}^{2,\alpha,\beta}(x) + \left(\tau_{n,q}^{1,\alpha,\beta}(x)\right)^2\right]\|g''\|_\infty.
\end{aligned}$$

Since $|L_{n,q}^{\alpha,\beta}(f,x)| \leq \|f\|_\infty$,

$$\begin{aligned}
|L_{n,q}^{\alpha,\beta}(f,x) - f(x)| &\leq \left|\tilde{L}_{n,q}^{\alpha,\beta}(f-g,x) - (f-g)(x)\right| + \left|\tilde{L}_{n,q}^{\alpha,\beta}(g,x) - g(x)\right| + \left|\frac{[n]_q\ x+\alpha}{[n]_q+\beta} - f(x)\right|\\
&\leq 2\|f-g\|_\infty + \left[\tau_{n,q}^{2,\alpha,\beta}(x) + \left(\tau_{n,q}^{1,\alpha,\beta}(x)\right)^2\right]\|g''\|_\infty + \omega\left(f, |\tau_{n,q}^{1,\alpha,\beta}(x)|\right).
\end{aligned}$$

Taking infimum overall $g \in C_B^2(\mathbb{R}^+)$, we get

$$|L_{n,q}^{\alpha,\beta}(f,x) - f(x)| \leq K_2\left(f,\ \tau_{n,q}^{2,\alpha,\beta}(x) + \left(\tau_{n,q}^{1,\alpha,\beta}(x)\right)^2\right) + \omega\left(f, |\tau_{n,q}^{1,\alpha,\beta}(x)|\right).$$

In view of $K_2(f\ ,\ \delta) \leq\ C\omega_2\left(f\ ,\ \sqrt{\delta}\right),\ \delta\ >\ 0$, we have

$$|L_{n,q}^{\alpha,\beta}(f,x) - f(x)| \leq C\omega_2\left(f, \sqrt{\tau_{n,q}^{2,\alpha,\beta}(x) + \left(\tau_{n,q}^{1,\alpha,\beta}(x)\right)^2}\right) + \omega\left(f, |\tau_{n,q}^{1,\alpha,\beta}(x)|\right),$$

which proves the Theorem 3.1.

4 Rate of convergence

In this section, we want to estimate the rate of convergence for the sequence of the operators $L_{n,q}^{\alpha,\beta}$. For any positive a, by

$$\overline{\omega}_a(f,\ \delta) = \sup_{|t-x|\leq\delta}\ \sup_{x,t\in[0,a]} |f(t) - f(x)|,$$

we denote the usual modulus of continuity of f on the closed interval $[0,\ a]$. Let $\rho(x) = 1 + \varphi^2(x)$, where $\varphi(x)$ is a monotone increasing continuous function on the real axis and B_ρ is the set of all functions f defined on the real axis satisfying the growth condition $|f(x)| \leq M_f\rho(x)$, where M_f is a constant depending only on f. Then B_ρ is a normed space with norm
$\|f\|_\rho = \sup\left\{\frac{|f(x)|}{\rho(x)} : x \in \mathbb{R}\right\}$, for any $f \in B_\rho$. Let C_ρ denote the subspace of all continuous function in B_ρ, and C_ρ^* the subspace of all function $f \in C_\rho$ for which $\lim_{|x|\to\infty}\left(\frac{f(x)}{\rho(x)}\right) = 0$.

Now, we give a rate of convergence theorem for the operator $L_{n,q}^{\alpha,\beta}$.

Theorem 4.1 *Let $f \in C_{\rho_0}(\mathbb{R}^+)$, $q = q_n \in (0,1)$ such that $q_n \to 1$ as $n \to \infty$ and $\overline{\omega}_{a+1}(f,\ \delta)$ be its modulus of continuity on the finite interval $[0,\ a+1] \subset \mathbb{R}^+$, where $a > 0$. Then*

$$\left\| L_{n,q}^{\alpha,\beta}(f) - f \right\|_{[0,a]} \le 6M_f\left(1+a^2\right)\tau_{n,q}^{2,\alpha,\beta}(x) + 2\overline{\omega}_{a+1}\left(f,\ \sqrt{\tau_{n,q}^{2,\alpha,\beta}(x)}\right).$$

Proof: For $x \in [0,a]$ and $t > a+1$, since $t - x > 1$, we have

$$|f(t) - f(x)| \le 6M_f\left(1+a^2\right)(t-x)^2. \tag{4.1}$$

For $x \in [0,a]$ and $t \le a+1$, we have

$$|f(t) - f(x)| \le \overline{\omega}_{a+1}(f,|t-x|) \le \left(1 + \frac{|t-x|}{\delta}\right)\overline{\omega}_{a+1}(f,\delta), \tag{4.2}$$

with $\delta > 0$. Form (4.1) and (4.2), we can write

$$|f(t) - f(x)| \le 6M_f\left(1+a^2\right)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\overline{\omega}_{a+1}(f,\delta),$$

for $x \in [0,a]$ and $t \ge 0$. Thus

$$\begin{aligned}\left| L_{n,q}^{\alpha,\beta}(f,x) - f(x) \right| &\le L_{n,q}^{\alpha,\beta}(|f(t) - f(x)|, x) \\ &\le 6M_f\left(1+a^2\right)L_{n,q}^{\alpha,\beta}\left((t-x)^2, x\right) \\ &\quad + \overline{\omega}_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\left(L_{n,q}^{\alpha,\beta}\left((t-x)^2, x\right)\right)^{\frac{1}{2}}\right).\end{aligned}$$

Hence, by Schwarz's inequality and Lemma 1.2, for every $q \in (0,1)$ and $x \in [0,\ a]$

$$\left| L_{n,q}^{\alpha,\beta}(f,x) - f(x) \right| \le 6M_f\left(1+a^2\right)\tau_{n,q}^{2,\alpha,\beta}(x) + \overline{\omega}_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\tau_{n,q}^{2,\alpha,\beta}(x)}\right).$$

By taking $\delta = \sqrt{\tau_{n,q}^{2,\alpha,\beta}(x)}$, we get the assertion of our theorem.

5 Weighted approximation

The Korovkin type theorems on weighted approximation of unbounded continuous functions on unbounded sets with single weight function were first proved by Gadzhiev [19, 20]. Now, we give Gadzhiev's results in weighted spaces.

Theorem 5.1 *(a) There exists a sequence of linear positive operators $A_n(C_\rho \to B_\rho)$ such that*

$$\lim_{n\to\infty} \|A_n(\varphi^v) - \varphi^v\|_\rho = 0\ ,\quad v = 0,\ 1,\ 2 \tag{5.1}$$

and a function $f^* \in C_\rho - C_\rho^*$ with $\lim_{n\to\infty} \|A_n(f^*) - f^*\|_\rho \ge 1$.
(b) If a sequence of linear positive operators $A_n(C_\rho \to B_\rho)$ such that satisfies conditions (4.1) then

$$\lim_{n\to\infty} \|A_n(f) - f\|_\rho = 0,$$

for every $f \in C_\rho^*$.

Theorem 5.2 *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For each $f \in C^*_{\rho_0}(\mathbb{R}^+)$, we have*

$$\lim_{n\to\infty} \|L^{\alpha,\beta}_{n,q_n}(f) - f\|_{\rho_0} = 0.$$

Proof: Using the theorem in [19], we see that it is sufficient to verify the following

$$\lim_{n\to\infty} \|L^{\alpha,\beta}_{n,q_n}(t^v,\ x) - x^v\|_{\rho_0} = 0\ ,\ \ v = 0,\ 1,\ 2\ . \tag{5.2}$$

Since $L^{\alpha,\beta}_{n,q_n}(1,x) = 1$, the first condition of (4.2) is satisfied for $v = 0$.

Now,

$$\begin{aligned}\left\|L^{\alpha,\beta}_{n,q_n}(t,\ x) - x\right\|_{\rho_0} &= \sup_{x\in(0,\infty)} \frac{|L^{\alpha,\beta}_{n,q_n}(t,\ x) - x|}{1+x^2} \\ &\leq \frac{\alpha}{[n]_{q_n}+\beta} + \frac{\beta}{[n]_{q_n}+\beta} \sup_{x\in(0,\infty)} \frac{x}{1+x^2} \\ &\leq \frac{\alpha+\beta}{[n]_{q_n}+\beta} \to 0 \ as\ n \to \infty\end{aligned}$$

and the second condition of (4.2) hold for $r = 1$.

Similarly, we have

$$\begin{aligned}\left\|L^{\alpha,\beta}_{n,q_n}\left(t^2,\ x\right) - x^2\right\|_{\rho_0} &= \sup_{x\in(0,\infty)} \frac{|L^{\alpha,\beta}_{n,q_n}\left(t^2,\ x\right) - x^2|}{1+x^2} \\ &\leq \left|\frac{q_n[n]^2_{q_n}}{\left([n]_{q_n}+\beta\right)^2} - 1\right| \sup_{x\in(0,\infty)} \frac{x^2}{1+x^2} \\ &\quad + \frac{(2\alpha+1+q_n)[n]_{q_n}}{\left([n]_{q_n}+\beta\right)^2} \sup_{x\in(0,\infty)} \frac{x}{1+x^2} + \frac{\alpha^2}{\left([n]_{q_n}+\beta\right)^2} \\ &\leq \left|\frac{(q_n-1)[n]^2_{q_n} - 2\beta[n]_{q_n} - \beta^2}{\left([n]_{q_n}+\beta\right)^2}\right| + \frac{(2\alpha+1+q_n)[n]_{q_n}}{\left([n]_{q_n}+\beta\right)^2} + \frac{\alpha^2}{\left([n]_{q_n}+\beta\right)^2} \\ &\leq \left|\frac{(q_n-1)[n]^2_{q_n}}{\left([n]_{q_n}+\beta\right)^2}\right| + \left|\frac{2\beta[n]_{q_n}}{\left([n]_{q_n}+\beta\right)^2}\right| + \left|\frac{\beta^2}{\left([n]_{q_n}+\beta\right)^2}\right| \\ &\quad + \frac{(2\alpha+1+q_n)[n]_{q_n}}{\left([n]_{q_n}+\beta\right)^2} + \frac{\alpha^2}{\left([n]_{q_n}+\beta\right)^2},\end{aligned}$$

which implies that

$$\left\|L^{\alpha,\beta}_{n,q_n}\left(t^2,\ x\right) - x^2\right\|_{\rho_0} \to 0\ \ as\ n \to \infty.$$

Thus the proof is completed.

We give the following theorem to approximate all functions in $C_{\rho_0}(\mathbb{R}^+)$. This type of results are given in [21] for locally integrable functions.

Theorem 5.3 *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For each $f \in C_{\rho_0}(\mathbb{R}^+)$ and $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in(0,\infty)} \frac{|L^{\alpha,\beta}_{n,q_n}(f,\ x) - f(x)|}{(1+x^2)^{1+\epsilon}} = 0.$$

Proof: For any fixed $x_0 > 0$,

$$\begin{aligned}\sup_{x\in(0,\infty)} \frac{|L_{n,q_n}^{\alpha,\beta}(f,\ x)-f(x)|}{(1+x^2)^{1+\epsilon}} &\le \sup_{x\le x_0} \frac{|L_{n,q_n}^{\alpha,\beta}(f,\ x)-f(x)|}{(1+x^2)^{1+\epsilon}} + \sup_{x\ge x_0} \frac{|L_{n,q_n}^{\alpha,\beta}(f,\ x)-f(x)|}{(1+x^2)^{1+\epsilon}} \\ &\le \left\| L_{n,q_n}^{\alpha,\beta}(f,\cdot)-f(\cdot)\right\|_{[0,x_0]} \\ &\quad +\|f\|_{x^2} \sup_{x\ge x_0} \frac{|L_{n,q_n}^{\alpha,\beta}\left(1+t^2,\ x\right)|}{(1+x^2)^{1+\epsilon}} + \sup_{x\ge x_0} \frac{|f(x)|}{(1+x^2)^{1+\epsilon}}.\end{aligned}$$

The first term of the above inequality tends to zero from Theorem 4.1. By Lemma 1.2 for any fixed $x_0 > 0$ it is easily seen that $\sup_{x\ge x_0} \frac{|L_{n,q_n}^{\alpha,\beta}\left(1+t^2,\ x\right)|}{(1+x^2)^{1+\epsilon}}$ tends to zero as $n \to \infty$. We can choose $x_0 > 0$ so large that the last part of the above inequality can be made small enough.
Thus the proof is completed.

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