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Some Labeling Techniques of Kusudama Flower Graph

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Abstract

This paper aims to focus on some labeling methods of Kusudama Flower Graph. We investigate kusudama flower graph with five types of labeling; Cordial, E-cordial, Prime, Total prime, Vertex prime.

Keywords: Graph Labeling, Kusudama Flower Graph, Cordial Labeling, Prime Labeling.
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1. Introduction

We begin with simple, finite, undirected graph $G=(V(G),E(G))$ where $V(G)$ and $E(G)$ denotes the vertex set and the edge set respectively. For a finite set A, $|A|$ denotes the number of elements of A. For all other terminology we follow Gross [5]. We provide some useful definitions for the present work.

Definition 1.1: The *graph labeling* is an assignment of numbers to the vertices or edges or both subject to certain condition(s).

A detailed survey of various graph labeling is explained in Gallian [4].

Definition 1.2: For a graph $G=(V(G),E(G))$, a mapping $f:V(G)\to\{0,1\}$ is called a *binary vertex labeling* of G and $f(v)$ is called the *label* of the vertex v of G under f. For an edge $e=uv$, the induced edge labeling $f^*:E(G)\to\{0,1\}$ defined as $f^*(uv)=|f(u)-f(v)|$.

Let $v_f(0)$ and $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$ and $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.3: A binary vertex labeling f of a graph G is called a *cordial labeling* if $|v_f(1)-v_f(0)|\le 1$ and $|e_f(1)-e_f(0)|\le 1$. A graph G is said to be *cordial* if it admits *cordial labeling*.

The concept of cordial labeling was introduced by Cahit [1]. Lee and Liu [6] proved that all complete bipartite graphs and all fans are cordial. Further, they proved that, the cycle C_n is cordial if and only if $n \not\equiv 2 \pmod 4$, the wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$, $n \geq 3$. Prajapati and Gajjar [8] proved that complement of wheel graph and complement of cycle graph are cordial if $n \not\equiv 4 \pmod 8$ or $n \not\equiv 7 \pmod 8$. Prajapati and Gajjar [9] proved that cordial labeling in the context of duplication of cycle graph and path graph.

Definition 1.4: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and let $f : E(G) \to \{0,1\}$. Define f^* on $V(G)$ by $f^* = \sum\{f(uv)/uv \in E(G)\}\ (mod\ \ 2)$. The function f is called an *E- cordial labeling* of G if $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. A graph is called *E-cordial* if it admits *E-cordial labeling*.

In 1997 Yilmaz and Cahit [14] introduced E-cordial labeling as a weaker version of edge-graceful labeling and with the blend of cordial labeling. They proved that the trees with n vertices, K_n, C_n are E-cordial if and only if $n \not\equiv 2 \pmod 4$ while $K_{m,n}$ admits E-cordial labeling if and only if $m+n \not\equiv 2 \pmod 4$.

Definition 1.5: A *prime labeling* of a graph G is an injective function $f : V(G) \to \{1,2,...,|V|\}$ such that for every pair of adjacent vertices u and v, $gcd(f(u), f(v)) = 1$. The graph which admits a *prime labeling* is called a *prime graph*.

The notion of a prime labeling was originated by Roger Entringer and was discussed in a paper by Tout et al. [11] . Many researchers have studied prime graphs. For e.g. Fu and Huang [3] have proved that P_n and $K_{1,n}$ are prime graphs. Lee et al. [7] have proved that W_n is a prime graph if and only if n is even. Vaidya and Prajapati [12] has proved that if $n_1 \geq 4$ is an even integer and n_2 is a natural number, then the graph obtained by identifying any of the rim vertices of a wheel W_{n_1} with an end vertex of a path graph P_{n_2} is a prime graph. Vaidya and Prajapati [13] have proved that switching the apex vertex in W_n is a prime graph and switching a rim vertex in W_n is a prime graph if $n+1$ is prime. In the same paper it has been proved that W_n is switching invariant if n is even.

Definition 1.6: G is called a *vertex prime graph* if there is a bijection $f : E(G) \to \{1,2,...,|E|\}$ such that for any vertex v, $\underset{uv \in E}{gcd}\{f(uv)\} = 1$. The bijection f is called a *vertex prime labeling* of G.

Definition 1.7: Let $G = (V,E)$ be a graph with p vertices and q edges. A bijection $f : V(G) \to \{1,2,...,|V|+|E|\}$ is said to be a *total prime labeling* if for each edge $e = uv$, the labels assigned to u and v are relatively prime and for each vertex of degree at least 2, the greatest common divisor of the labels on the incident edges is 1. A graph which admits total prime labeling is called *total prime graph.*

Prime labeling and vertex prime labeling are introduced in [11] and [2]. Combining these two, The notion of a total prime labeling was originated by Ramasubramanian and Kala [10] have proved that paths P_n, star $K_{1,n}$, bistar, comb, cycles C_n where n is even, helm H_n, $K_{2,n}$ and fan graph are total prime graph.

In this paper, for every natural number n the set $\{1,2,...,n\}$ will be denoted by $[n]$.

We present a new graph inspired from the famous Japanese Kusudama art namely Kusudama Flower. Kusudama is a paper model that is made from a number of identical origami shapes that are glued or sewn together through their points to form a spherical shape. The word origami comes from two Japanese words: ori, which means to fold, and kami, which means paper. Origami models are made strictly by folding paper with no cutting or gluing involved.

Definition 1.8: Let v_0 be the apex vertex and $v_1, v_2, v_3, \ldots, v_{2n-1}, v_{2n}$ be consecutive $2n$ rim vertices of wheel graph W_{2n}, $n \geq 3$. Subdivide spoke edge $v_0 v_{2i-1}$ with vertex w_i and at each w_i, join two copies of path of length 2; $P_2^l = v_0, u_{2i-1}, w_i$ and $P_2^r = v_0, u_{2i}, w_i$, for each $i \in [n]$. The resulting graph is called *kusudama flower graph KF_n*.

Example:

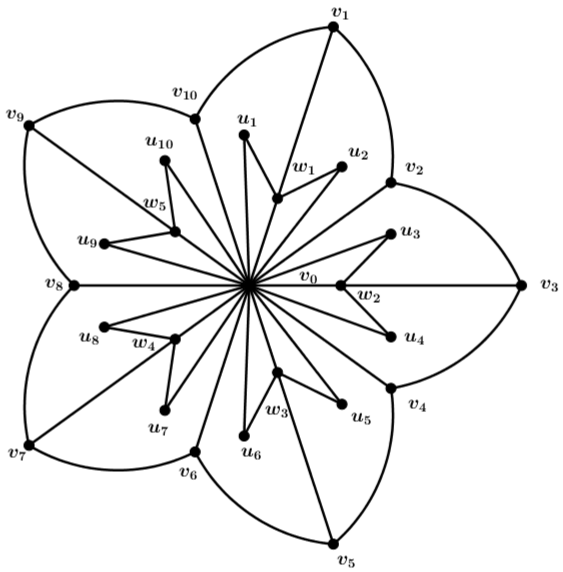

Figure 1: The kususdama flower graph KF_5.

2. Main Results

Theorem 2.1: KF_n is cordial.

Proof: For the graph KF_n, $V(KF_n) = \{v_0, w_i / 1 \leq i \leq n\} \cup \{u_i, v_i / 1 \leq i \leq 2n\}$ and
$E(KF_n) = \{v_0 v_{2i}, v_0 w_i, v_0 u_{2i-1}, v_0 u_{2i}, w_i v_{2i-1}, w_i u_{2i-1}, w_i u_{2i} / 1 \leq i \leq n\} \cup \{v_i v_{i+1} / 1 \leq i \leq 2n-1\} \cup \{v_{2n} v_1\}$. Therefore $|V(KF_n)| = 5n+1$ and $|E(KF_n)| = 9n$. Define $f : V(KF_n) \rightarrow \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = v_0; \\ 1 & \text{if } x = u_i,\ i \in [2n]; \\ 0 & \text{if } x \in \{w_i, v_{2i-1}\},\ i \in [n]; \\ \frac{1+(-1)^i}{2} & \text{if } x = v_{2i},\ i \in [n]. \end{cases}$$

Thus $v_f(1) = 2n + \left\lceil \frac{n+1}{2} \right\rceil$, $v_f(0) = 2n + \left\lfloor \frac{n+1}{2} \right\rfloor$. The induced edge labeling $f^* : E(KF_n) \rightarrow \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore $e_f(1) = 4n + \left\lfloor \frac{n}{2} \right\rfloor$, $e_f(0) = 4n + \left\lfloor \frac{n}{2} \right\rfloor$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$ for cordial labeling. So, f admits cordial labeling of KF_n. Hence KF_n is cordial graph.

Theorem 2.2: KF_n is E-cordial.

Proof: For the graph KF_n, $V(KF_n) = \{v_0, w_i/1 \le i \le n\} \cup \{u_i, v_i/1 \le i \le 2n\}$ and
$E(KF_n) = \{v_0v_{2i}, v_0w_i, v_0u_{2i-1}, v_0u_{2i}, w_iv_{2i-1}, w_iu_{2i-1}, w_iu_{2i}/1 \le i \le n\} \cup \{v_iv_{i+1}/1 \le i \le 2n-1\} \cup \{v_{2n}v_1\}$. Therefore $|V(KF_n)| = 5n+1$ and $|E(KF_n)| = 9n$. Using parity of n, we have the following cases:

case 1: $n \equiv 0,3 \pmod 4$. Define $f : E(KF_n) \to \{0,1\}$ as follows:

$$f(e) = \begin{cases} 1 & \text{if } e \in \{v_0u_{2i-1}, v_0w_i, u_{2i-1}w_i, w_iv_{2i-1}\},\ i \in [n]; \\ 0 & \text{if } e \in \{v_0u_{2i}, u_{2i}w_i, v_{2i-1}v_{2i}\},\ i \in [n]; \\ \frac{1+(-1)^{i+1}}{2} & \text{if } e = v_0v_{2i},\ i \in [n]; \\ 0 & \text{if } e = v_{2i}v_{2i+1},\ i \in [n-1]; \\ 0 & \text{if } e = v_nv_1. \end{cases}$$

Thus $e_f(1) = 4n + \left\lceil \frac{n}{2} \right\rceil$, $e_f(0) = 4n + \left\lfloor \frac{n}{2} \right\rfloor$. The induced vertex labeling $f^* : V(KF_n) \to \{0,1\}$ is $f^*(v) = \sum\{f(uv)/uv \in E(KF_n)\}(\ mod\ \ 2)$. Therefore $v_f(0) = 2n + \left\lceil \frac{n+1}{2} \right\rceil$, $v_f(1) = 2n + \left\lceil \frac{n}{2} \right\rceil$.

case 2: $n \equiv 1,2 \pmod 4$. Define $f : E(KF_n) \to \{0,1\}$ as follows:

$$f(e) = \begin{cases} 1 & \text{if } e \in \{v_0w_i, u_{2i-1}w_i, w_iv_{2i-1}\},\ i \in [n]; \\ 0 & \text{if } e \in \{v_0u_{2i}, u_{2i}w_i, v_{2i-1}v_{2i}, v_0u_{2i-1}\},\ i \in [n]; \\ \frac{1+(-1)^{i+1}}{2} & \text{if } e = v_0v_{2i},\ i \in [n]; \\ 1 & \text{if } e = v_{2i}v_{2i+1},\ i \in [n-1]; \\ 1 & \text{if } e = v_nv_1. \end{cases}$$

Thus $e_f(1) = 4n + \left\lceil \frac{n}{2} \right\rceil$,$e_f(0) = 4n + \left\lfloor \frac{n}{2} \right\rfloor$. The induced vertex labeling $f^* : V(KF_n) \to \{0,1\}$ is $f^*(v) = \sum\{f(uv)/uv \in E(KF_n)\}(\ mod\ \ 2)$. Therefore $v_f(0) = 2n + \left\lceil \frac{n}{2} \right\rceil$, $v_f(1) = 2n + \left\lceil \frac{n+1}{2} \right\rceil$.

Therefore from both the cases f satisfies the conditions $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$ for E-cordial labeling. So, f admits E-cordial labeling of KF_n. Hence KF_n is E-cordial graph.

Theorem 2.3: KF_n is prime graph.

Proof: For the graph KF_n, $V(KF_n) = \{v_0, w_i/1 \le i \le n\} \cup \{u_i, v_i/1 \le i \le 2n\}$ and
$E(KF_n) = \{v_0v_{2i}, v_0w_i, v_0u_{2i-1}, v_0u_{2i}, w_iv_{2i-1}, w_iu_{2i-1}, w_iu_{2i}/1 \le i \le n\} \cup \{v_iv_{i+1}/1 \le i \le 2n-1\} \cup \{v_{2n}v_1\}$.

Therefore $|V(KF_n)| = 5n+1$ and $|E(KF_n)| = 9n$. Define $f : V(KF_n) \to [5n+1]$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = v_0; \\ 5i - \left(2 + \frac{1-(-1)^i}{2}\right) & \text{if } x = u_{2i-1},\ i \in [n]; \\ 5i - \left(\frac{1-(-1)^i}{2}\right) & \text{if } x = u_{2i},\ i \in [n]; \\ 5i - \left(1 + \frac{1-(-1)^i}{2}\right) & \text{if } x = w_i,\ i \in [n]; \\ 5i - \left(3 \cdot \frac{1+(-1)^i}{2}\right) & \text{if } x = v_{2i-1},\ i \in [n]; \\ 5i+1 & \text{if } x = v_{2i},\ i \in [n]. \end{cases}$$

Clearly f is an injective function. $gcd(f(v), f(u)) = 1$ for every pair of adjacent vertices u and v of KF_n. Then f admits prime labeling of KF_n. Hence KF_n is a prime graph.

Theorem 2.4: KF_n is vertex prime graph.

Proof: For the graph KF_n, $V(KF_n) = \{v_0, w_i/1 \le i \le n\} \cup \{u_i, v_i/1 \le i \le 2n\}$ and
$E(KF_n) = \{v_0v_{2i}, v_0w_i, v_0u_{2i-1}, v_0u_{2i}, w_iv_{2i-1}, w_iu_{2i-1}, w_iu_{2i}/1 \le i \le n\} \cup \{v_iv_{i+1}/1 \le i \le 2n-1\} \cup \{v_{2n}v_1\}$.
Therefore $|V(KF_n)| = 5n+1$ and $|E(KF_n)| = 9n$. Define $f : E(KF_n) \to [9n]$ as follows:

$$f(x) = \begin{cases} 9i-8 & \text{if } x = v_0u_{2i-1},\ i \in [n]; \\ 9i-7 & \text{if } x = u_{2i-1}w_i,\ i \in [n]; \\ 9i-6 & \text{if } x = v_0w_i,\ i \in [n]; \\ 9i-5 & \text{if } x = v_0u_{2i},\ i \in [n]; \\ 9i-4 & \text{if } x = u_{2i}w_i,\ i \in [n]; \\ 9i-3 & \text{if } x = w_iv_{2i-1},\ i \in [n]; \\ 9i-2 & \text{if } x = v_{2i-1}v_{2i},\ i \in [n]; \\ 9i & \text{if } x = v_{2i}v_{2i+1},\ i \in [n-1]; \\ 9i-1 & \text{if } x = v_0v_{2i},\ i \in [n]; \\ 9n & \text{if } x = v_{2n}v_1. \end{cases}$$

So, f is a bijection and $\underset{uv \in E}{gcd}\{f(uv)\} = 1$ for each vertex $v \in V(KF_n)$. The edges are labeled such that for any vertex v_i, the greatest common divisor of labels of all the edges incident with v_i is 1. Then f admits vertex prime labeling of KF_n. Hence KF_n is a vertex prime graph.

Theorem 2.5: KF_n is total prime graph.

Proof: For the graph KF_n, $V(KF_n) = \{v_0, w_i/1 \le i \le n\} \cup \{u_i, v_i/1 \le i \le 2n\}$ and
$E(KF_n) = \{v_0v_{2i}, v_0w_i, v_0u_{2i-1}, v_0u_{2i}, w_iv_{2i-1}, w_iu_{2i-1}, w_iu_{2i}/1 \le i \le n\} \cup \{v_iv_{i+1}/1 \le i \le 2n-1\} \cup \{v_{2n}v_1\}$.

Therefore $|V(KF_n)| = 5n+1$ and $|E(KF_n)| = 9n$. Define $f : V(KF_n) \cup E(KF_n) \to [14n+1]$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = v_0; \\ 5i - \left(2 + \frac{1-(-1)^i}{2}\right) & \text{if } x = u_{2i-1},\ i \in [n]; \\ 5i - \left(\frac{1-(-1)^i}{2}\right) & \text{if } x = u_{2i},\ i \in [n]; \\ 5i - \left(1 + \frac{1-(-1)^i}{2}\right) & \text{if } x = w_i,\ i \in [n]; \\ 5i - \left(3 \cdot \frac{1+(-1)^i}{2}\right) & \text{if } x = v_{2i-1},\ i \in [n]; \\ 5i+1 & \text{if} x = v_{2i},\ i \in [n]; \\ 5n+9i-7 & \text{if } x = v_0u_{2i-1},\ i \in [n]; \\ 5n+9i-6 & \text{if } x = u_{2i-1}w_i,\ i \in [n]; \\ 5n+9i-5 & \text{if } x = v_0w_i,\ i \in [n]; \\ 5n+9i-4 & \text{if } x = v_0u_{2i},\ i \in [n]; \\ 5n+9i-3 & \text{if } x = u_{2i}w_i,\ i \in [n]; \\ 5n+9i-2 & \text{if } x = w_iv_{2i-1},\ i \in [n]; \\ 5n+9i-1 & \text{if } x = v_{2i-1}v_{2i},\ i \in [n]; \\ 5n+9i+1 & \text{if } x = v_{2i}v_{2i+1},\ i \in [n-1]; \\ 5n+9i & \text{if } x = v_0v_{2i},\ i \in [n]; \\ 14n+1 & \text{if } x = v_{2n}v_1. \end{cases}$$

So, f is a bijection. According to this pattern, the vertices are labeled such that for any edge $e = uv \in KF_n$, $\gcd(f(u), f(v)) = 1$. Also the edges are labeled such that for any vertex v_i, the greatest common divisor of labels of all the edges incident with v_i is 1. Then f admits total prime labeling of KF_n. Hence KF_n is total prime graph.

3. Conclusion

we have derived five new results by investigating some labeling techniques of kusudama flower graph. More exploration is possible for other graph families in the context of different graph labeling problems.

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