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Solution of Nonlinear Partial Differential Equation Using Generalized Functional Separable Method

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Abstract

In this paper second order partial differential equation having nonlinear source term is solved by using generalized functional separable method. The generalized travelling wave solution has been obtained for special type of functional equation that arises most frequently in many engineering and scientific problems.

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1. Introduction

Many nonlinear partial differential equations with quadratic or power nonlinearities,

$$f_1(x)g_1(y)\Pi_1[w]+f_2(x)g_2(y)\Pi_2[w]+.....+f_m(x)g_m(y)\Pi_m[w]=0 \tag{1}$$

have exact solutions of the form (2). Where $\Pi_i[w]$ are differential forms that are the products of nonnegative integer powers of the function *w* and its partial derivatives $\partial_x w, \partial_y w, \partial_{xx} w, \partial_{xy} w, \partial_{yy} w, \partial_{xxx} w,$ etc.

Linear separable equations of mathematical physics admit exact solutions in the form

$$w(x,y)=\phi_1(x)\psi_1(y)+\phi_2(x)\psi_2(y)+.....+\phi_n(x)\psi_n(y) \tag{2}$$

This can be referred as generalized separable solutions of (1).

On substituting expression (2) in to the differential equation (1), one arrives at a functional differential equation [5]

$$\Phi_1(X)\Psi_1(Y)+\Phi_2(X)\Psi_2(Y)+.....+\Phi_k(X)\Psi_k(Y)=0 \text{ for the } \phi_i(x) \text{ and } \psi_i(y). \tag{3}$$

The functionals $\Phi_j(X)$ and $\Psi_j(Y)$ depend only on *x* and *y* respectively,

$$\left.\begin{aligned}\Phi_j(X)&\equiv\Phi_j(x,\phi,\phi_1',\phi_1'',......,\phi_n,\phi_n',\phi_n'')\\ \Psi_j(Y)&\equiv\Psi_j(y,\psi,\psi_1',\psi_1'',......,\psi_n,\psi_n',\psi_n'')\end{aligned}\right\} \tag{4}$$

The formulas are written out for the case of a second order equation (1).

2. Solution of Functional Differential Equations (3) – (4)

Dividing the equation by Ψ_k provided $\Psi_k \neq 0$ and differentiating with respect to *y*, we obtain a similar equation but with fewer terms

$$\widetilde{\Phi}_1(X)\widetilde{\Psi}_1(Y)+.....+\widetilde{\Phi}_{k-1}(X)\widetilde{\Psi}_{k-1}(Y)=0\,,\ \widetilde{\Phi}_j=\Phi_j(X)\,,\ \widetilde{\Psi}_j=\left[\Psi_j(Y)/\Psi_k(Y)\right]'_y \tag{5}$$

We continue the above procedure until we obtain a simple separable two term equation

$$\widehat{\Phi}_1(X)\widehat{\Psi}_1(Y)+\widehat{\Phi}_2(X)\widehat{\Psi}_2(Y)=0 \tag{6}$$

Three cases must be considered.

Nondegenerate case: $\left|\widehat{\Phi}_1(X)\right|+\left|\widehat{\Phi}_2(X)\right| \not\equiv 0$ and $\left|\widehat{\Psi}_1(Y)\right|+\left|\widehat{\Psi}_2(Y)\right| \not\equiv 0$.

Then equation (6) is equivalent to the ordinary differential equations

$\widehat{\Phi}_1(X)+C\widehat{\Phi}_2(X)=0$, $C\widehat{\Psi}_1(Y)-\widehat{\Psi}_2(Y)=0$, where C is an arbitrary constant. The equations $\widehat{\Phi}_2=0$ and $\widehat{\Psi}_1=0$ correspond to the limit case $C=\infty$.

Two degenerate cases:

$\widehat{\Phi}_1(X)\equiv 0,\ \widehat{\Phi}_2(X)\equiv 0 \ \Rightarrow\ \widehat{\Psi}_{1,2}(Y)$ are any;

$\widehat{\Psi}_1(Y)\equiv 0,\ \widehat{\Psi}_2(Y)\equiv 0 \ \Rightarrow\ \widehat{\Phi}_{1,2}(X)$ are any.

While reducing the number of terms in the functional-differential equation (3) – (4) by differentiation, redundant constants of integration arises. These constants must be "removed" at the final stage. Furthermore, the resulting equation can be of a higher-order than the original equation. To avoid these difficulties, it is convenient to reduce the solution of the functional-differential equation to the solution of a bilinear functional equation of a standard form and solution of a system of ordinary differential equations. Thus, the original problem splits into two simpler problems which are explained in two stages as follows:

At the first stage, we treat the functional differential equation (3) as a purely bilinear functional equation that depends on *x* and *y* where $\Phi_n=\Phi_n(X)$ and $\Psi_n=\Psi_n(Y)$ are unknowns. (*n* = 1, 2, ...,*k*)

Equation (3) has $k-1$ different solutions [3] as mentioned below:

$$\left.\begin{aligned}\Phi_i(X)&=C_{i,1}\Phi_{m+1}(X)+C_{i,2}\Phi_{m+2}(X)+.....+C_{i,k-m}\Phi_k(X), && i=1,.....,m;\\ \Psi_{m+j}(Y)&=-C_{1,j}\Psi_1(Y)-C_{2,j}\Psi_2(Y)-.....-C_{m,j}\Psi_m(Y), && j=1,.....,k-m;\end{aligned}\right\} \tag{7}$$

$m=1,2,.....,k-1;$

wherethe $C_{i,j}$ are arbitrary constants.

The functions $\Phi_{m+1}(X),.....,\Phi_k(X),\Psi_1(Y),.....,\Psi_m(Y)$ on the right-hand sides of equation in (7) are defined arbitrarily. It is apparent that for fixed *m*, solution (7) contains $m(k-m)$ arbitrary constants.

At the second stage, we successively substitute the $\Phi_i(X)$ and $\Psi_j(Y)$ of (4) into all solutions (7) to obtain systems of ordinary differential equations for the unknown functions $\phi_p(x)$ and $\psi_p(y)$. Solving these systems, we get generalized separable solutions of the form (1).

Case – I

Consider the functional equation

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 = 0 \text{ Where } \Phi_i = \Phi_i(X) \text{ and } \Psi_i = \Psi_i(Y),\ i = 1,2,3 \tag{8}$$

To find the solution of equation (8),

Putting $i = 1$ and $m = 2$ in first equation of (7), we get $\Phi_1 = C_{1,1}\Phi_3$

Putting $i = 2$ and $m = 2$ in first equation of (7), we get $\Phi_2 = C_{2,1}\Phi_3$

Putting $j = 1$ and $m = 2$ in second equation of (7), we get $\Psi_3 = -C_{1,1}\Psi_1 - C_{2,1}\Psi_2$

Therefore, the first solution is

$\Phi_1 = A_1\Phi_3,\ \Phi_2 = A_2\Phi_3,\ \Psi_3 = -A_1\Psi_1 - A_2\Psi_2$ where $A_1 = C_{1,1}$ and $A_2 = C_{2,1}$

Putting $m = 1$ and $j = 2$ in second equation of (7), we get $\Psi_1 = -\dfrac{1}{C_{1,2}}\Psi_3$

Putting $m = 1$ and $j = 1$ in second equation of (7), we get $\Psi_2 = -C_{1,1}\Psi_1 = \dfrac{C_{1,1}}{C_{1,2}}\Psi_3$

Now, $\Phi_3 = -\dfrac{\Psi_1}{\Psi_3}\Phi_1 - \dfrac{\Psi_2}{\Psi_3}\Phi_2 = \dfrac{1}{C_{1,2}}\Phi_1 - \dfrac{C_{1,1}}{C_{1,2}}\Phi_2$

Therefore, the second solution is

$\Psi_1 = A_1\Psi_3,\ \Psi_2 = A_2\Psi_3,\ \Phi_3 = -A_1\Phi_1 - A_2\Phi_2$ Where $A_1 = -1/C_{1,2}$ and $A_2 = C_{1,1}/C_{1,2}$

Therefore, the two solutions of equation (8) are,

$$\left.\begin{array}{l}\Phi_1 = A_1\Phi_3,\ \Phi_2 = A_2\Phi_3,\ \Psi_3 = -A_1\Psi_1 - A_2\Psi_2 \\ \Psi_1 = A_1\Psi_3,\ \Psi_2 = A_2\Psi_3,\ \Phi_3 = -A_1\Phi_1 - A_2\Phi_2\end{array}\right\} \tag{9}$$

The functions on the right-hand sides of the equations in (9) are assumed to be arbitrary.

Case – II

Consider the functional equation

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 + \Phi_4\Psi_4 = 0 \text{ ,Where } \Phi_i = \Phi_i(X) \text{ and } \Psi_i = \Psi_i(Y),\ i = 1,2,3,4 \tag{10}$$

To find the solution of equation (10),

Putting $i = 1$ and $m = 2$ in first equation of (7), we get $\Phi_1 = C_{1,1}\Phi_3 + C_{1,2}\Phi_4$

Putting $i = 2$ and $m = 2$ in first equation of (7), we get $\Phi_2 = C_{2,1}\Phi_3 + C_{2,2}\Phi_4$

Putting $j=1$ and $m=2$ in second equation of (7), we get $\Psi_3=-C_{1,1}\Psi_1-C_{2,1}\Psi_2$

Putting $j=2$ and $m=2$ in second equation of (7), we get $\Psi_4=-C_{1,2}\Psi_1-C_{2,2}\Psi_2$

Therefore, the first solution dependent on four arbitrary constants is

$\Phi_1=A_1\Phi_3+A_2\Phi_4$, $\Phi_2=A_3\Phi_3+A_4\Phi_4$, $\Psi_3=-A_1\Psi_1-A_3\Psi_2$, $\Psi_4=-A_2\Psi_1-A_4\Psi_2$

Where $A_1=C_{1,1}$, $A_2=C_{1,2}$, $A_3=C_{2,1}$ and $A_4=C_{2,2}$

Putting $i=1$ and $m=3$ in first equation of (7), we get $\Phi_1=C_{1,1}\Phi_4$

Putting $i=2$ and $m=3$ in first equation of (7), we get $\Phi_2=C_{2,1}\Phi_4$

Putting $i=3$ and $m=3$ in first equation of (7), we get $\Phi_3=C_{3,1}\Phi_4$

Putting $j=1$ and $m=3$ in first equation of (7), we get $\Psi_4=-C_{1,1}\Psi_1-C_{2,1}\Psi_2-C_{3,1}\Psi_3$

Therefore, the second solution involving three arbitrary constants is

$\Phi_1=A_1\Phi_4$, $\Phi_2=A_2\Phi_4$, $\Phi_3=A_3\Phi_4$, $\Psi_4=-A_1\Psi_1-A_2\Psi_2-A_3\Psi_3$

Where $A_1=C_{1,1}$, $A_2=C_{2,1}$ and $A_3=C_{3,1}$

Putting $m=1$ and $j=3$ in second equation of (7), we get $\Psi_1=-\dfrac{1}{C_{1,3}}\Psi_4$

Putting $m=1$ and $j=1$ in second equation of (7), we get $\Psi_2=\dfrac{C_{1,1}}{C_{1,3}}\Psi_4$

Putting $m=1$ and $j=2$ in second equation of (7), we get $\Psi_3=\dfrac{C_{1,2}}{C_{1,3}}\Psi_4$

Now, $\Phi_4=\dfrac{1}{C_{1,3}}\Phi_1-\dfrac{C_{1,1}}{C_{1,3}}\Phi_2-\dfrac{C_{1,2}}{C_{1,3}}\Phi_3$

Therefore, the other second solution involving three arbitrary constants is

$\Psi_1=A_1\Psi_4$, $\Psi_2=A_2\Psi_4$, $\Psi_3=A_3\Psi_4$, $\Phi_4=-A_1\Phi_1-A_2\Phi_2-A_3\Phi_3$

Where $A_1=-1/C_{1,3}$, $A_2=C_{1,1}/C_{1,3}$, and $A_3=C_{1,2}/C_{1,3}$

Therefore, the three solutions of equation (10) are,

$$\left.\begin{array}{l}\Phi_1=A_1\Phi_3+A_2\Phi_4,\Phi_2=A_3\Phi_3+A_4\Phi_4,\Psi_3=-A_1\Psi_1-A_3\Psi_2,\Psi_4=-A_2\Psi_1-A_4\Psi_2\\ \Phi_1=A_1\Phi_4,\Phi_2=A_2\Phi_4,\Phi_3=A_3\Phi_4,\Psi_4=-A_1\Psi_1-A_2\Psi_2-A_3\Psi_3\\ \Psi_1=A_1\Psi_4,\Psi_2=A_2\Psi_4,\Psi_3=A_3\Psi_4,\Phi_4=-A_1\Phi_1-A_2\Phi_2-A_3\Phi_3\end{array}\right\}\qquad(11)$$

3. Example

Consider the equation $\dfrac{\partial w}{\partial t}=a(t)\dfrac{\partial^2 w}{\partial x^2}+b(t)\dfrac{\partial w}{\partial x}+c(t)F(w)$ (12)

where $F(w)$ represents nonlinear source term. [3]

4. Functional separable solution:

We look for functional separable solution of the special form

$$w = w(z),\ \ z = \phi(t)x + \psi(t) \tag{13}$$

which is known as a generalized travelling wave solution.

Our purpose is to determine the unknown functions $w(z),\ \phi(t),\ \psi(t)$ and $F(w)$.

Substituting (13) in to (12), we get

$$w'_z \cdot z'_t = a\frac{\partial}{\partial x}\left[w'_z \cdot z'_x\right] + b\left[w'_z \cdot z'_x\right] + cF(w)$$

On simplifying,

$$\phi'_t \cdot x + \psi'_t = a\phi^2 \frac{w''_{zz}}{w'_z} + b\phi + c\frac{F(w)}{w'_z} \tag{14}$$

To obtain a functional differential equation with two variables, substituting $x = \frac{z-\psi}{\phi}$ into equation (14),

$$\phi'_t\left[\frac{z-\psi}{\phi}\right] + \psi'_t = a\phi^2 \frac{w''_{zz}}{w'_z} + b\phi + c\frac{F(w)}{w'_z}$$

This gives

$$-\psi'_t + \frac{\psi}{\phi}\phi'_t - \frac{\phi'_t z}{\phi} + a\phi^2 \frac{w''_{zz}}{w'_z} + b\phi + c\frac{F(w)}{w'_z} = 0$$

This can be treated as the functional equation

$$\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \phi_4\psi_4 = 0$$

Where ϕ_i and ψ_i $(i = 1,\ 2,\ 3,\ 4)$ can be defined from Case – II which are mentioned below.

$$\left.\begin{aligned} &\phi_1 = -\psi'_t + \frac{\psi}{\phi}\phi'_t + b\phi,\ \ \phi_2 = -\frac{\phi'_t}{\phi},\ \ \phi_3 = a\phi^2,\ \ \phi_4 = c \\ &\psi_1 = 1,\ \ \psi_2 = z,\ \ \psi_3 = \frac{w''_{zz}}{w'_z},\ \ \psi_4 = \frac{F(w)}{w'_z} \end{aligned}\right\} \tag{15}$$

Putting these expressions (15) into first relations of (11), we get the system of ordinary differential equations (16) – (19).

$$-\psi'_t + \frac{\psi}{\phi}\phi'_t + b\phi = A_1 a\phi^2 + A_2 c \tag{16}$$

$$-\frac{\phi'_t}{\phi} = A_3 a\phi^2 + A_4 c \tag{17}$$

$$\frac{w''_{zz}}{w'_z} = -A_1 - A_3 z \tag{18}$$

$$\frac{F(w)}{w'_z} = -A_2 - A_4 z \tag{19}$$

where $A_1, \ldots, A_4$ are arbitrary constants.

Now we consider two possibilities for A_4 :

(i) When $A_4 \neq 0$

Reducing equation (16) into simplified form,

$$-\frac{\partial}{\partial t}\left(\frac{\psi}{\phi}\right)+b=A_1 a\phi+\frac{A_2 c}{\phi}$$

Integrating, we get the solution for $\psi(t)$

$$\psi(t)=-\phi(t)\left[-bt+A_1 a\int\phi(t)\,dt+A_2 c\int\frac{dt}{\phi(t)}+c_1\right]$$

Equation of (17) can be written as,

$$\frac{d\phi}{dt}+A_4 c\phi=-A_3 a\phi^3$$

For more simplification, taking $\phi^{-2}=v$, we get

$\frac{dv}{dt}-2vA_4 c=2A_3 a$ which is first order linear differential equation.

Now, integrating factor is $e^{-2A_4 ct}$

Solution given by,

$v\cdot e^{-2A_4 ct}=\int 2A_3 a\cdot e^{-2A_4 ct}\,dt+c_2$ which gives

$$\phi^{-2}=-\frac{A_3 a}{A_4 c}+c_2 e^{2A_4 ct}$$

Therefore, the solution for $\phi(t)$ can be written as

$$\phi(t)=\pm\left[c_2 e^{2A_4 ct}-\frac{A_3 a}{A_4 c}\right]^{-\frac{1}{2}}$$

On integrating equation (18), we get

$$\log w'_z=-A_1 z-A_3\frac{z^2}{2}+c_3$$

On simplifying, we get

$$w'_z=e^{\left(-\frac{1}{2}A_3 z^2-A_1 z\right)+c_3}$$

Again integrating,

$$w(z)=c_3\int\exp\left(-\frac{1}{2}A_3 z^2-A_1 z\right)+c_4$$

Using equation (19), we have

$$F(w) = w'_z\left[-(A_2 + A_4 z)\right]$$

Substituting $w'_z = e^{\left(-\frac{1}{2}A_3 z^2 - A_1 z\right)+c_3}$ in to above equation, we get

$$F(w) = -(A_2 + A_4 z)\left[c_3 \exp\left(-\frac{1}{2}A_3 z^2 - A_1 z\right)\right]$$

Thus for $A_4 \neq 0$ the solution of system of ordinary differential equations (16) – (19) is given by

$$\left.\begin{aligned}
\psi(t) &= -\phi(t)\left[-bt + A_1 a\int \phi(t)\,dt + A_2 c\int \frac{dt}{\phi(t)} + c_1\right] \\
\phi(t) &= \pm\left[c_2 e^{2A_4 ct} - \frac{A_3 a}{A_4 c}\right]^{-\frac{1}{2}} \\
w(z) &= c_3\int \exp\left(-\frac{1}{2}A_3 z^2 - A_1 z\right) + c_4 \\
F(w) &= -(A_2 + A_4 z)\left[c_3 \exp\left(-\frac{1}{2}A_3 z^2 - A_1 z\right)\right]
\end{aligned}\right\} \tag{20}$$

where $c_1, \ldots\ldots, c_4$ are arbitrary constants.

Substituting the expression for $w(z)$, $\phi(t)$, $\psi(t)$ and $F(w)$ in equation (13), we get the generalized travelling wave solution.

The dependence $F = F(w)$ is defined by the last two relations in parametric form.

In special case $A_3 = c_4 = 0$, $A_1 = -1$ and $c_3 = 1$ the source function can be represented in explicit form as

$$F(w) = -w(A_4 \ln w + A_2) \tag{21}$$

(ii) When $A_4 = 0$

Reducing (17) into simplified form,

$$-\phi^{-3}d\phi = A_3 a\,dt$$

Integrating we get solution for $\phi(t)$,

$$\phi(t) = \pm\frac{1}{\sqrt{2A_3 at + c_1}} \tag{22}$$

Using equation (16),

$$\psi(t) = -\phi(t)\left[-bt + A_1 a\int \phi(t)\,dt + A_2 c\int \frac{dt}{\phi(t)} + c_2\right]$$

Therefore we get,

$$\psi(t)=\frac{1}{\sqrt{2A_3at+c_1}}\left[-bt-A_1a\int\left(2A_3at+c_1\right)^{-\frac{1}{2}}dt-A_2c\int\left(2A_3at+c_1\right)^{\frac{1}{2}}dt+c_2\right]$$

On simplifying, we get

$$\psi(t)=\frac{-bt}{\sqrt{2A_3at+c_1}}-\frac{A_1}{A_3}-\frac{A_2c}{3A_3a}\left(2A_3t+c_1\right)+\frac{c_2}{\sqrt{2A_3t+c_1}} \qquad (23)$$

and the solutions to the other equations are determined by the last formulas in (20) where $A_4=0$.

5. Conclusion

In present study we have discussed the Generalized Functional Separable Method for the solution of nonlinear partial differential equation. The method is applied to solve partial differential (12) having nonlinear source term. The solution of equation (12) consists of two possibilities for arbitrary constant A_4. In case (i) When $A_4\neq 0$, the solution is obtained in equation (20) and in case (ii) when $A_4=0$, the solution obtained in equation (22) and (23). Further, this method can be extended to solve higher order nonlinear partial differential equation.

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