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Cross-Correlation and Auto-Correlation Theorems for Fractional Fourier Transform

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Abstract

Fractional Fourier transform (FRFT) is a generalization of classical Fourier transform which has received considerable attention of researchers since last four decades. It is now established as one of the most useful modern mathematical tools. It has been applied in almost every field of science and engineering specifically in signal processing, signal analysis, optical communication and quantum mechanics etc. Many definitions of FRFT already exist. Recently a new definition of FRFT of real order α has been introduced using Mittag-Leffler function in fractional calculus environment. In this paper correlation and auto-correlation theorems of FRFT have been established analytically in the fractional calculus environment. These properties can be employed to produce some better applications in various fields. Fractional Fourier transform (FRFT) of some particular functions also have been derived analytically.

Keywords: Mittag-Leffeler function, Fractional Integral, Fourier transform and fractional Fourier transform.
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1. Introduction

Integral transforms have been used as mathematical tools and produced variety of applications in almost every area of science and engineering, since their inception. Fractionalization of a transform is a new area of research and many integral transforms have been fractionalized till date, e.g., fractional Fourier transform, fractional Laplace etc. [2, 5, 6, 7, 8 and 14].

The fractional Fourier transform (FRFT) was originated by Wiener in 1929 [19]. Subsequently many other scientists defined fractional Fourier transforms differently and developed their theoretical properties [8]. After a long period Namias presented its consolidated definition in 1980 [14]. Many researchers followed Namias's work and produced enormous applications of FRFT in almost every field of science and engineering including signal processing, signal analysis, optical communication and quantum mechanics [1, 3, 8, 9, 11, 12, 14 and 15].

Recently a novel definition of FRFT of real order α has been introduced by Jumarie using Mittag-Leffler function [5]. This transform plays the same role for the fractional derivatives as Fourier transform does for the ordinary derivatives. Particularly for α = 1 this FRFT reduces to the Fourier transform in usual sense. Hence it is better suited for the definition of FRFT as compared to the other definitions proposed earlier in the literature.

2. Preliminaries

2.1. Fractional Riemann-Liouville Integral

Let *f(x)* be a piecewise continuous function on (0,∞) and integrable on any finite sub interval of [0,∞) then for $t > 0$, Riemann-Liouville fractional integral of *f(x)* of order α is defined as [4, 13, 16 and 17].

$$ {}_aD_t^{-\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_a^t (t-\xi)^{\alpha-1} f(\xi)d\xi \qquad (2.1)$$

where Re$(\alpha) > 0$

2.2. The Integral with respect to $(du)^\alpha$:

The integral with respect to $(du)^\alpha$ is defined as the solution of the fractional differential equation [5-6]

$$dv = g(u)(du)^\alpha,\ u \geq 0 \text{ and } v(0) = 0 \qquad (2.2)$$

which is provided by the following lemma [5-6]

Lemma 2.1: If *g(u)* is a continuous function, then the solution of equality (2.2) is defined by following equality

$$v = \int_0^u g(\xi)\,(d\xi)^\alpha = \alpha\int_0^u (u-\xi)^{\alpha-1} g(\xi)\ d\xi, \quad 0 < \alpha \leq 1 \qquad (2.3)$$

2.3. Mittag – Leffler Function:

In 1903, the Swedish mathematician Gosta Mittag-Leffler defined a function [10] as follows

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)} \qquad (2.4)$$

where $\alpha > 0$ and $z \in C$. The Mittag-Leffler function is an extension of exponential function and reduces to exponential function for $\alpha = 1$.

Remark 2.1: Following identity holds for $\lambda \in \mathrm{C}$ [5]

$$E_\alpha(\lambda x^\alpha)E_\alpha(\lambda y^\alpha) = E_\alpha(\lambda(x+y)^\alpha),\quad \lambda \in \mathrm{C} \qquad (2.5)$$

2.4. Fourier Transform: The Fourier transform of a function *f(x)* is defined as [18]

$$\mathcal{F}\,[f(x)] = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx \qquad (2.6)$$

provided that the integral of (2.6) converges.

2.5. Fractional Fourier Transform (FRFT)

Jumarie [5] presented the definition of fractional Fourier transform (FRFT) by using Mittag -Leffler function as follows

$$\mathcal{F}_\alpha[f(x)] = \hat{f}_\alpha(\omega) = \int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha) f(x)(dx)^\alpha, \quad 0 < \alpha < 1 \tag{2.7}$$

2.6. Inversion formula for FRFT

If $\hat{f}_\alpha(\omega)$ is the FRFT of $f(x)$, then its inversion formula is defined as [5]

$$f(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i(-\omega x)^\alpha)\, \hat{f}_\alpha(\omega)\, (d\omega)^\alpha, \quad 0 < \alpha < 1 \tag{2.8}$$

where M_α is the period of the complex-valued Mittag -Leffler function and it satisfies the following equality $E_\alpha[i(M_\alpha)^\alpha] = 1$.

2.7. Convolution of Fractional Order

The convolution of order α of the two functions $f(x)$ and $g(x)$ is defined as [6]

$$[(f * g)(x)]_\alpha = \int_{-\infty}^{+\infty} f(\omega)\, g(x-\omega)\, (d\omega)^\alpha \tag{2.9}$$

2.8. The Fractalization of *sinx* and *cosx*

The Fractalization of *sinx* and *cosx* is defined in slightly different form as follows [5]

$$E_\alpha(ix^\alpha) = cos_\alpha x^\alpha + i\, sin_\alpha x^\alpha \tag{2.10}$$

with $cos_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k\alpha}}{(2k\alpha!)}$

and $sin_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{[(2k+1)\alpha]!}$

3. Dirac Delta Function of Fractional Order

Definition 3.1: We define a function

$$\delta_\alpha(x, \varepsilon) = \begin{cases} 0, & x \notin [0, \varepsilon] \\ \frac{\varepsilon^{-\alpha}}{2}, & 0 < x < \varepsilon \end{cases} \tag{3.1}$$

In the limiting case, when $\varepsilon \to 0$ we have the limit

$$lim_{\varepsilon \to 0}\, \delta_\alpha(x, \varepsilon) = \delta_\alpha(x)$$

which is provided by the following lemma

Lemma 3.1: If $\delta_\alpha(x - a)$ is a function of fractional order α, $0 < \alpha \leq 1$ then following formula holds

$$\int_{-\infty}^{\infty} f(x)\, \delta_\alpha(x - a)\, (dx)^\alpha = \alpha f(a) \tag{3.2}$$

Proof: Taking left hand side of (3.2), we have

$$\text{LHS} = \int_{a-\varepsilon}^{a+\varepsilon} f(x)\, \delta_\alpha(x - a)\, (dx)^\alpha$$

$$= \alpha \int_{a-\varepsilon}^{a+\varepsilon} (a + \varepsilon - \omega)^{\alpha-1} f(\omega)\, \delta_\alpha(\omega - a)\, d\omega$$

$$= \alpha \int_{a-\varepsilon}^{a+\varepsilon} \varepsilon^{\alpha-1} f(a)\, \delta_\alpha(\omega, \varepsilon)\, d\omega$$

$$= \alpha \varepsilon^{\alpha-1} f(a) \frac{\varepsilon^{-\alpha}}{2}\, 2\varepsilon$$

Using (3.1) we get the desired result.

Remark 3.1: As particular case, we have [5]

$$\int_{-\infty}^{\infty} f\,(x)\,\delta_\alpha(x)\,(dx)^\alpha \;=\; \alpha f\,(0) \qquad (3.3)$$

Example 3.1: We can easily obtain fractional Fourier transform (FRFT) of Dirac delta function of fractional order by using (3.2) as follows

$$\mathcal{F}_\alpha\{\,\delta_\alpha(x-a)\} \;=\; \int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha)\;\delta_\alpha(x-a)\;(dx)^\alpha$$
$$=\;\alpha E_\alpha(i\omega^\alpha a^\alpha) \qquad (3.4)$$

As particular case, we have $\mathcal{F}_\alpha\{\delta_\alpha(x)\} = \alpha$

i.e. $\alpha = \int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha)\,\delta_\alpha(x)\;(dx)^\alpha$ (3.5)

3.1 Relation between Fractional Dirac delta and Mittag- Leffler function:

The relation between $\delta_\alpha(x-a)$ and $E_\alpha(x^\alpha)$ is presented in following lemma, which can be established directly using (3.5)

Lemma 3.2: The following result holds

$$\frac{\alpha}{(M_\alpha)^\alpha}\int_{-\infty}^{+\infty} E_\alpha(-i\omega^\alpha(x-a)^\alpha)\;(d\omega)^\alpha = \delta_\alpha(x-a) \qquad (3.6)$$

where M_α satisfy the equality $E_\alpha[i(\,M_\alpha)^\alpha] = 1$ and is called period of the Mittag- Leffler function.

Remark 3.2: *(i)* Equation (3.6) can also be written as

$$\frac{\alpha}{(M_\alpha)^\alpha}\int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha(x+a)^\alpha)\;(d\omega)^\alpha = \delta_\alpha(x+a) \qquad (3.7)$$

(ii) As particular case equation (3.6) and (3.7) reduce to following equalities [5]

$$\frac{\alpha}{(M_\alpha)^\alpha}\int_{-\infty}^{+\infty} E_\alpha(-i\omega^\alpha x^\alpha)\;(d\omega)^\alpha = \delta_\alpha(x) \qquad (3.8)$$

and

$$\frac{\alpha}{(M_\alpha)^\alpha}\int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha)\;(d\omega)^\alpha = \delta_\alpha(x) \qquad (3.9)$$

Remark 3.2: Using (3.8), we can obtain inverse fractional Fourier transform of some elementary functions as follows

(i) ${\mathcal{F}_\alpha}^{-1}\{1\} = \frac{1}{\alpha}\delta_\alpha(x)$

(ii) ${\mathcal{F}_\alpha}^{-1}\{E_\alpha(i(a\omega)^\alpha)\} = \frac{1}{\alpha}\delta_\alpha(x+a)$

(iii) ${\mathcal{F}_\alpha}^{-1}\{cos_\alpha(a\omega)^\alpha\} = \frac{1}{2\alpha}[\delta_\alpha(x+a)+\delta_\alpha(x-a)]$

(iv) ${\mathcal{F}_\alpha}^{-1}\{sin_\alpha(a\omega)^\alpha\} = \frac{1}{2\alpha}i[\delta_\alpha(x-a)-\delta_\alpha(x+a)]$

where a is any positive integer.

4. Fractional Fourier Transform of Some Elementary Functions

In this section, we have been evaluated Fractional Fourier Transform of some elementary functions which are as follows

Example 4.1: let us consider a function

$$f(x) = \begin{cases} x^k\,, & if\ 0 < x \le x_0 \\ 0\,, & otherwise \end{cases}$$

where k denotes a positive integer.

Then fractional Fourier Transform of above function is given by

$\mathcal{F}_\alpha\{x^k\} = \int_0^{x_0} E_\alpha(i\omega^\alpha x^\alpha)\ x^k\ (dx)^\alpha$

Using (2.3), we have

$$\mathcal{F}_\alpha\{x^k\} = \Gamma(\alpha+1){x_0}^{\alpha+k} \sum_{m=0}^{+\infty} \frac{(i\omega^\alpha {x_0}^\alpha)^m}{\Gamma(\alpha m+1)} \times \frac{\Gamma(\alpha m+k+1)}{\Gamma(\alpha m+k+1+\alpha)} \tag{4.1}$$

Special Case 4.1: Substituting $k = 0$ in (4.1), we have

$$\mathcal{F}_\alpha\{1\} = \Gamma(\alpha+1){x_0}^\alpha\ E_{\alpha,(\alpha+1)}(\mathrm{i}\omega^\alpha {x_0}^\alpha) \tag{4.2}$$

where $E_{\alpha,\beta}(\mathrm{z}) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m+\beta)}$ is the generalization of Mittag-Leffler function and $Re(\alpha) > 0$, $Re(\beta) > 0$ and $z \in C$. It reduces into Mittag - Leffler function for $\beta = 1$.

Remark 4.1:

Equation (4.2) can also be expressed as the term of Mittag-Leffler function, therefore we have the following equality

$$\mathcal{F}_\alpha\{1\} = \Gamma(\alpha+1)\left[\frac{E_\alpha(i\omega^\alpha {x_0}^\alpha)-1}{i\omega^\alpha}\right] \tag{4.3}$$

Further, we can extend the result (4.3) as

$$\mathcal{F}_\alpha\{E_\alpha(ic^\alpha x^\alpha)\} = \Gamma(\alpha+1)\left[\frac{E_\alpha(i(\omega+c)^\alpha {x_0}^\alpha)-1}{i(\omega+c)^\alpha}\right] \tag{4.4}$$

where c is a positive integer.

and on equating real and imaginary parts of both sides of (4.4), we have

$$\mathcal{F}_\alpha\{cos_\alpha(cx)^\alpha\} = \Gamma(\alpha+1)\left[\frac{sin_\alpha((\omega+c)^\alpha {x_0}^\alpha)}{(\omega+c)^\alpha}\right] \tag{4.5}$$

and $\mathcal{F}_\alpha\{sin_\alpha(cx)^\alpha\} = [1 - cos_\alpha((\omega+c)^\alpha {x_0}^\alpha)]$ (4.6)

where equations (4.5) and (4.6) represent the fractional Fourier transform of the sine and cosine functions of fractional order.

Example4.2: let us consider a function

$$f(x) = \begin{cases} E_\alpha(i\omega^\alpha c^\alpha), & if\ |x| < x_0 \\ 0, & otherwise \end{cases}$$

which is a constant function, then following result holds

$$\mathcal{F}_\alpha\{E_\alpha(i\omega^\alpha c^\alpha)\} = \Gamma(\alpha+1)\ (2x_0)^\alpha\ E_{\alpha,(\alpha+1)}(i\ 2^\alpha \omega^\alpha {x_0}^\alpha)$$

5. Cross-correlation and auto-correlation theorems for fractional Fourier transform

This section contains Cross-correlation and auto-correlation theorems of fractional Fourier transform where cross-correlation and auto-correlation are defined in the fractional environment by (5.2) and (5.3)

Definition 5.1: We define a cross-correlation ($\bullet$) of two functions f and g as follows

$$[(f \bullet g)(x)\ ]_\alpha = [\ \bar{f}[-(x)] * g(x)\ ]_\alpha \tag{5.1}$$

where $*$ denotes convolution of fractional order defined in (2.9)

Therefore (5.1) takes the form

$$[(f \bullet g)(x)\ ]_\alpha = \int_{-\infty}^{\infty} \bar{f}[(\omega)]\, g(x+\omega)(d\omega)^\alpha, \qquad 0 < \alpha \le 1 \tag{5.2}$$

provided the integral (5.2) converges.

Remark 5.1: Substituting $f = g$ in (5.2) then auto-correlation of fractional order α, $0 < \alpha \le 1$ is defined as follows

$$[(f \bullet f)(x)\ ]_\alpha = \int_{-\infty}^{\infty} \bar{f}[(\omega)]\, f(x+\omega)(d\omega)^\alpha,\ 0 < \alpha \le 1 \tag{5.3}$$

provided the integral (5.3) converges.

Theorem 5.1: If $\hat{f}_\alpha(\omega)$ denotes fractional Fourier transform, $\overline{Z}$ is the complex conjugate,

$$f(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i(-\omega x)^\alpha)\, \hat{f}_\alpha(\omega)\, (d\omega)^\alpha \tag{5.4}$$

and

$$g(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i(-\omega x)^\alpha)\, \hat{g}_\alpha(\omega)\, (d\omega)^\alpha \tag{5.5}$$

then following equality holds

$$(f \bullet g)(x) = \frac{1}{(M_\alpha)^\alpha} \mathcal{F}_\alpha[\overline{\hat{f}_\alpha(\omega)}\, \hat{g}_\alpha(\omega)] \tag{5.6}$$

where $(f \bullet g)$ is given by (5.2).

Proof: Using (5.1), we have

$$(f \bullet g)(x) = \int_{-\infty}^{+\infty} \overline{f(\xi)} g(x+\xi)(d\xi)^\alpha \tag{5.7}$$

Using (5.4) and (5.5) in (5.7), we obtain

$$(f \bullet g)(x) = \int_{-\infty}^{+\infty} (d\xi)^\alpha \left\{\frac{1}{(M_\alpha)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i(\omega\xi)^\alpha)\, \hat{f}_\alpha(\omega)(d\omega)^\alpha\right\} \times$$

$$\left\{\frac{1}{(M_\alpha)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i(-[\omega'((x+\xi)]^\alpha)\, \hat{g}_\alpha(\omega')\, (d\omega')^\alpha\right\}$$

$$= \frac{1}{(M_\alpha)^{2\alpha}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{f}_\alpha(\omega)\hat{g}_\alpha(\omega')\{E_\alpha(-i(\omega'-\omega)^\alpha\xi^\alpha)\} \times$$

$$\{E_\alpha(-i(\omega' x)^\alpha)\}(d\xi)^\alpha(d\omega)^\alpha(d\omega')^\alpha$$

$$= \frac{1}{(M_\alpha)^{2\alpha}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{f}_\alpha(\omega)\, \hat{g}_\alpha(\omega') E_\alpha(-i(\omega' x)^\alpha)\, (d\omega)^\alpha(d\omega')^\alpha \times$$

$$\int_{-\infty}^{+\infty} E_\alpha(-i(\omega'-\omega)^\alpha\xi^\alpha)(d\xi)^\alpha$$

Using (3.8) and (3.2) respectively, we obtain the result (5.6).

Auto – correlation theorem is a special case of cross-correlation theorem, which is given as follows

Theorem 5.2: If $\hat{f}_\alpha(\omega)$ denotes fractional Fourier transform, $\overline{Z}$ is the complex conjugate and

$$f(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i(-\omega x)^\alpha)\, \hat{f}_\alpha(\omega)\, (d\omega)^\alpha \tag{5.8}$$

then following equality holds

$$(f \bullet f)(x) = \frac{1}{(M_\alpha)^\alpha} \mathcal{F}_\alpha\left[\left|\ \hat{f}_\alpha(\omega)\right|^2\right]$$

Theorem 5.3: If $\hat{f}_\alpha(\omega)$ is the fractional Fourier transform of $f(x)$ then fractional Fourier transform of $f(x)\sin_\alpha(ax)^\alpha$ is $\frac{1}{2i}\hat{f}_\alpha(\omega+c)^\alpha$ - $\frac{1}{2i}\hat{f}_\alpha(\omega-c)^\alpha$, $0 < \alpha \le 1$ and $\omega > 0$

where c is any arbitrary constant.

Proof: we have by definition of fractional Fourier transform

$\mathcal{F}_\alpha[f(x)\ sin_\alpha(cx)^\alpha] = \int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha)\ f(x)\ sin_\alpha(cx)^\alpha\ (dx)^\alpha$

$= \int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha) f(x) \left\{\frac{[E_\alpha(i(cx)^\alpha) - E_\alpha(-i(cx)^\alpha)]}{2i}\right\} (dx)^\alpha$ [By using (2.10)]

$= \frac{1}{2i}\int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha)\ E_\alpha(i(cx)^\alpha) f(x)\ (dx)^\alpha - \frac{1}{2i}\int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha)\ E_\alpha(-i(cx)^\alpha) f(x)\ (dx)^\alpha$

Therefore, we have the result.

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Conclusion

Fractional Fourier transform is now established as the most extensively used tool in almost every area of science and technology. Fractionalization of Fourier transform via Mittag-Leffler function is a novel concept. This concept opens a new and wide area of research to produce variety of applications. In the present paper correlation and auto-correlation theorems of fractional Fourier transform have been derived analytically. Fractional Fourier transform (FRFT) of some particular functions also have been derived analytically. This paper adds one step to develop the mathematical theory of fractional Fourier transform in the fractional calculus environment. These theorems can be applied in signal processing and signal analysis specifically in pattern recognition and electron tomography where the cross-correlation and auto-correlation concept are used.

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