Mathematics Today Vol.34 (December 2018) 125-143
ISSN 0976-3228, e-ISSN 2455-9601

IDENTITIES INVOLVING k-Fibonacci AND k-Lucas SEQUENCES

ASHOK DNYANDEO GODASE

ABSTRACT. In this paper, we investigate the different binomial sums of k-Fibonacci and k-Lucas sequences. Furthermore, we obtain congruence properties of k-Fibonacci and k-Lucas sequences.

1. INTRODUCTION

The well known Fibonacci sequence is an integer sequence, which is defined by the numbers that satisfy the second order recurrence relation $\mathscr{F}_n = \mathscr{F}_{n-1} + \mathscr{F}_{n-2}$ with the initial conditions $\mathscr{F}_0 = 0$ and $\mathscr{F}_1 = 1$. Fibonacci numbers have many interesting properties and applications in various research areas such as Architecture, Engineering, Nature and Art. The Lucas sequence is companion sequence of Fibonacci sequence defined with the Lucas numbers which are defined with the recurrence relation $\mathscr{L}_n = \mathscr{L}_{n-1} + \mathscr{L}_{n-2}$ with the initial conditions $\mathscr{L}_0 = 2$ and $\mathscr{L}_1 = 1$. Binet's formulas for the Fibonacci and Lucas numbers are

$$\mathscr{F}_n = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} \qquad \text{and} \qquad \mathscr{L}_n = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$. The positive root r_1 is known as the golden ratio.

The Fibonacci and Lucas sequences are generalised by changing the initial conditions or changing the recurrence relation. One of the famous generalization of the Fibonacci sequence is k- Fibonacci sequence first introduced by Falcon and Plaza in [5]. The k- Fibonacci sequence is defined by the numbers which satisfy the second order recurrence relation $\mathscr{F}_{k,n} = k\mathscr{F}_{k,n-1} + \mathscr{F}_{k,n-2}$ with the initial conditions $\mathscr{F}_{k,0} = 0$ and $\mathscr{F}_{k,1} = 1$. Falcon [6] defined the k- Lucas sequence which is companion sequence of k- Fibonacci sequence defined with the k- Lucas numbers which are defined with the recurrence relation $\mathscr{L}_{k,n} = k\mathscr{L}_{k,n-1} + \mathscr{L}_{k,n-2}$ with the initial conditions $\mathscr{L}_{k,0} = 2$ and $\mathscr{L}_{k,1} = k$. Binet's formulas for the k- Fibonacci and k- Lucas numbers are

$$\mathscr{F}_{k,n} = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} \qquad \text{and} \qquad \mathscr{L}_{k,n} = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $r_2 = \frac{k-\sqrt{k^2+4}}{2}$ are the roots of the characteristic equation $x^2 - kx - 1 = 0$. The characteristic roots r_1 and r_2 satisfy the properties

$$r_1 - r_2 = \sqrt{k^2+4} = \sqrt{\delta}, \quad r_1 + r_2 = k, \quad r_1 r_2 = -1.$$

1991 *Mathematics Subject Classification.* Primary 11B39; Secondary 11B37.

Key words and phrases. Fibonacci Sequence, Lucas Sequence, k-Fibonacci Sequence, k-Lucas Sequence.

The reader can refer to [4–12] for properties and applications of k- Fibonacci and k- Lucas numbers. Some well known properties of k- Fibonacci and k- Lucas numbers are listed below.

(1) $\mathscr{F}_{k,n-r}\mathscr{F}_{k,n+r} - {\mathscr{F}_{k,n}}^2 = (-1)^{n+1-r}{\mathscr{F}_{k,r}}^2$ **(Catalan's Identity)**,

(2) $\mathscr{F}_{k,n-1}\mathscr{F}_{k,n+1} - {\mathscr{F}_{k,n}}^2 = (-1)^n$ **(Cassini's Identity)** ,

(3) $\mathscr{F}_{k,r+1}\mathscr{F}_{k,n} = (-1)^n\mathscr{F}_{k,r-n}$ **(d'Ocagene's Identity)** ,

(4) $\mathscr{F}_{k,r}\mathscr{F}_{k,n+1} + \mathscr{F}_{k,r-1}\mathscr{F}_{k,n} = \mathscr{F}_{k,n+r}$ **(Convolution Theorem)** ,

(5) $\lim_{n\to\infty}\frac{\mathscr{F}_{k,n}}{\mathscr{F}_{k,n-r}} = r_1^r$ **(Asymptotic Behaviour)** .

The generating functions for the subsequence of k-Fibonacci and k-Lucas sequences are

(6) $$\sum_{n=0}^{\infty}\mathscr{F}_{k,tn}x^n = \frac{x\mathscr{F}_{k,t}}{1 - xL_{k,t} + x^2(-1)^t},$$

(7) $$\sum_{n=0}^{\infty}\mathscr{L}_{k,tn}x^n = \frac{2 - x\mathscr{L}_{k,t}}{1 - xL_{k,t} + x^2(-1)^t},$$

(8) $$\sum_{n=0}^{\infty}\mathscr{F}_{k,tn}\mathscr{L}_{k,tn}x^n = \frac{x\mathscr{F}_{k,2t}}{1 - xL_{k,2t} + x^2(-1)^{2t}}.$$

The sums of k-Fibonacci and k-Lucas sequences are

(9) $$\sum_{i=1}^{n}\mathscr{F}_{k,i} = \frac{\mathscr{F}_{k,n+1} + \mathscr{F}_{k,n} - (2+k)}{k},$$

(10) $$\sum_{i=1}^{n}\mathscr{L}_{k,i} = \frac{\mathscr{L}_{k,n+1} + \mathscr{L}_{k,n} - 1}{k},$$

(11) $$\sum_{i=1}^{n}\mathscr{F}_{k,2i} = \frac{\mathscr{F}_{k,2n+1} - 1}{k},$$

(12) $$\sum_{i=1}^{n}\mathscr{F}_{k,2i-1} = \frac{\mathscr{F}_{k,2n}}{k},$$

(13) $$\sum_{i=1}^{n}\mathscr{L}_{k,2i} = \frac{\mathscr{L}_{k,2n+1} - k}{k},$$

(14) $$\sum_{i=1}^{n}\mathscr{L}_{k,2i-1} = \frac{\mathscr{L}_{k,2n} - 2}{k},$$

(15) $$\sum_{i=1}^{n}\mathscr{F}_{k,i}^2 = \frac{\mathscr{F}_{k,n+1}\mathscr{F}_{k,n} - k^2}{k},$$

(16) $$\sum_{i=1}^{n}\mathscr{L}_{k,i}^2 = \frac{k\mathscr{L}_{k,n+1}\mathscr{L}_{k,n} + k^2}{k},$$

(17) $$\sum_{i=1}^{n}\mathscr{F}_{k,i}\mathscr{L}_{k,i} = \frac{\mathscr{F}_{k,2n+1} - 1}{k}.$$

In this paper, we have adapted the techniques of Carlitz [2] and Zhizheng Zhang [3] to k-Fibonacci and k-Lucas sequences and derived many interesting binomial and congruence identities for k-Fibonacci and k-Lucas sequences.

2. The Main Results

In this section, we explore certain binomial properties of the k-Fibonacci and k-Lucas sequences.

Lemma 2.1. *Let $u = r_1$ or r_2, then*

(a) $u^2 = ku + 1$.
(b) $u^n = u\mathscr{F}_{k,n} + \mathscr{F}_{k,n-1}$.
(c) $u^{2n} = u^n\mathscr{L}_{k,n} - (-1)^n$.
(d) $u^{tn} = u^n\dfrac{\mathscr{F}_{k,tn}}{\mathscr{F}_{k,n}} - (-1)^n - \dfrac{\mathscr{F}_{k,(t-1)n}}{\mathscr{F}_{k,n}}$.
(e) $u^{sn}\mathscr{F}_{k,rn} - u^{rn}\mathscr{F}_{k,sn} = (-1)^{sn}\mathscr{F}_{k,(r-s)n}$.

Theorem 2.2. *For $n, r, s, t \geq 1$, we have*

(a) $\mathscr{F}_{k,n+t} = \mathscr{F}_{k,n}\mathscr{F}_{k,t+1} + \mathscr{F}_{k,n-1}\mathscr{F}_{k,t}$.
(b) $\mathscr{F}_{k,2n+t} = \mathscr{L}_{k,n}\mathscr{F}_{k,n+t} - (-1)^n\mathscr{F}_{k,t}$.
(c) $\mathscr{F}_{k,sn+t} = \dfrac{\mathscr{F}_{k,sn}}{\mathscr{F}_{k,n}}\mathscr{F}_{k,n+t} - (-1)^n\dfrac{\mathscr{F}_{k,(s-1)n}}{\mathscr{F}_{k,n}}\mathscr{F}_{k,t}$.
(d) $\mathscr{F}_{k,sn+t}\mathscr{F}_{k,rn} - \mathscr{F}_{k,rn+t}\mathscr{F}_{k,sn} = (-1)^{sn}\mathscr{F}_{k,t}\mathscr{F}_{k,(r-s)n}$.

Theorem 2.3. *For $n, r, s, t \geq 1$ and $\mathscr{D}_{k,n} = \mathscr{F}_{k,n}$ or $\mathscr{L}_{k,n}$, we have*

(1) $\mathscr{D}_{k,2n} = \sum_{i=0}^{n}\binom{n}{i}k^i\mathscr{D}_{k,i}$.
(2) $\mathscr{D}_{k,2n+t} = \sum_{i=0}^{n}\binom{n}{i}k^i\mathscr{D}_{k,i+t}$.
(3) $\mathscr{D}_{k,rn+t} = \sum_{i=0}^{n}\binom{n}{i}\mathscr{F}_{k,r}^{i}\mathscr{F}_{k,r-1}^{n-i}\mathscr{D}_{k,i+t}$.
(4) $\mathscr{D}_{k,2rn+t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}\mathscr{L}_{k,r}^{i}\mathscr{D}_{k,ri+t}$.
(5) $\mathscr{D}_{k,trn+l} = \dfrac{1}{\mathscr{F}_{k,r}^{n}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}\mathscr{F}_{k,(t-1)r}^{n-i}\mathscr{F}_{k,tr}^{i}\mathscr{D}_{k,ri+l}$.
(6) $\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\mathscr{D}_{k,r(n-i)+i+t}\mathscr{F}_{k,r}^{i} = \mathscr{D}_{k,t}\mathscr{F}_{k,r-1}^{n}$.
(7) $\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\mathscr{D}_{k,ri+t}\mathscr{F}_{k,r-1}^{(n-i)} = \mathscr{D}_{k,n+t}\mathscr{F}_{k,r}^{n}$.
(8) $\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\mathscr{F}_{k,sm}^{(n-i)}\mathscr{F}_{k,rm}^{(i)}\mathscr{D}_{k,m[rn+i(s-r)]+t} = (-1)^{smn}\mathscr{D}_{k,t}\mathscr{F}_{k,(r-s)m}^{n}$.

Lemma 2.4. *Let $u = r_1$ or r_2, then*

(1) $k + (k^2+1)u = u^3$.
(2) $1 + ku + u^6 = \mathscr{L}_{k,2}u^4$.
(3) $1 + ku + u^{10} = \mathscr{L}_{k,4}u^6$.
(4) $1 + ku + u^{18} = \mathscr{L}_{k,8}u^{10}$.
(5) $1 + ku + u^{34} = \mathscr{L}_{k,16}u^{18}$.
(6) $1 + ku + u^{66} = \mathscr{L}_{k,32}u^{34}$.
(7) $1 + ku + u^{130} = \mathscr{L}_{k,64}u^{66}$.
(8) $1 + ku + u^{258} = \mathscr{L}_{k,128}u^{130}$.
(9) $1 + ku + u^{514} = \mathscr{L}_{k,256}u^{258}$.
(10) $1 + ku + u^{1026} = \mathscr{L}_{k,512}u^{514}$.
(11) $1 + ku + u^{2050} = \mathscr{L}_{k,1024}u^{1026}$.

In general, if $\mathscr{L}_{k,n}$ is n^{th} k-Lucas sequence and $u = r_1$ or r_2, then

$$1 + ku + u^{2(2^{n+1}+1)} = \mathscr{L}_{k,2^{n+1}} u^{2(2^n+1)}.$$

Theorem 2.5. *For $t \geq 1$ and $\mathscr{D}_{k,n} = \mathscr{F}_{k,n}$ or $\mathscr{L}_{k,n}$, we have*

(1) $\mathscr{D}_{k,t+3} = (k^2+1)\mathscr{D}_{k,t+1} + k\mathscr{D}_{k,t}$.

(2) $\mathscr{D}_{k,t+4} = \dfrac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+6}}{\mathscr{L}_{k,2}}$.

(3) $\mathscr{D}_{k,t+6} = \dfrac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+10}}{\mathscr{L}_{k,4}}$.

(4) $\mathscr{D}_{k,t+10} = \dfrac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+18}}{\mathscr{L}_{k,8}}$.

(5) $\mathscr{D}_{k,t+18} = \dfrac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+34}}{\mathscr{L}_{k,16}}$.

(6) $\mathscr{D}_{k,t+34} = \dfrac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+66}}{\mathscr{L}_{k,32}}$.

(7) $\mathscr{D}_{k,t+66} = \dfrac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+130}}{\mathscr{L}_{k,64}}$.

(8) $\mathscr{D}_{k,t+130} = \dfrac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+258}}{\mathscr{L}_{k,128}}$.

(9) $\mathscr{D}_{k,t+258} = \dfrac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+514}}{\mathscr{L}_{k,256}}$.

(10) $\mathscr{D}_{k,t+514} = \dfrac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+1026}}{\mathscr{L}_{k,512}}$.

(11) $\mathscr{D}_{k,t+1026} = \dfrac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+2050}}{\mathscr{L}_{k,1024}}$.

In general, for $t \geq 1$, we have

$$\mathscr{D}_{k,t+2^{n+1}+2} = \frac{\mathscr{D}_{k,t} + k\mathscr{D}_{k,t+1} + \mathscr{D}_{k,t+2^{n+2}+2}}{\mathscr{L}_{k,2^{n+1}}}.$$

Theorem 2.6. *For $n, t \geq 1$ and $\mathscr{D}_{k,n} = \mathscr{F}_{k,n}$ or $\mathscr{L}_{k,n}$, we have*

(1) $\mathscr{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,2}{}^{i} \mathscr{D}_{k,4i+6j+t}$.

(2) $\mathscr{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,4}{}^{i} \mathscr{D}_{k,6i+10j+t}$.

(3) $\mathscr{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,8}{}^{i} \mathscr{D}_{k,10i+18j+t}$.

(4) $\mathscr{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,16}{}^{i} \mathscr{D}_{k,18i+34j+t}$.

(5) $\mathscr{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,32}{}^{i} \mathscr{D}_{k,34i+66j+t}$.

(6) $\mathscr{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,64}{}^{i} \mathscr{D}_{k,66i+130j+t}$.

(7) $\mathscr{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,128}{}^{i} \mathscr{D}_{k,130i+258j+t}$.

(8) $\mathscr{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,256}{}^{i} \mathscr{D}_{k,258i+514j+t}$.

(9) $\mathscr{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,512}{}^{i} \mathscr{D}_{k,514i+1026j+t}$.

(10) $\mathscr{D}_{k,n+t} = \sum\limits_{i+j+s=n} \dbinom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,1024}{}^{i} \mathscr{D}_{k,1026i+2050j+t}$.

In general, for $r,n,t \geq 1$, we have

$$\mathscr{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} \mathscr{L}_{k,2^{r+1}}{}^{i} \mathscr{D}_{k,2^{r+1}(i+2j)+2(i+j)+t}.$$

Theorem 2.7. *For $n,t \geq 1$ and $\mathscr{D}_{k,n} = \mathscr{F}_{k,n}$ or $\mathscr{L}_{k,n}$, we have*

(1) $\mathscr{D}_{k,6n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,2}{}^{i} \mathscr{D}_{k,4i+j+t}$.

(2) $\mathscr{D}_{k,10n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,4}{}^{i} \mathscr{D}_{k,6i+j+t}$.

(3) $\mathscr{D}_{k,18n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,8}{}^{i} \mathscr{D}_{k,10i+j+t}$.

(4) $\mathscr{D}_{k,34n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,16}{}^{i} \mathscr{D}_{k,18i+j+t}$.

(5) $\mathscr{D}_{k,66n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,32}{}^{i} \mathscr{D}_{k,34i+j+t}$.

(6) $\mathscr{D}_{k,130n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,64}{}^{i} \mathscr{D}_{k,66i+j+t}$.

(7) $\mathscr{D}_{k,258n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,128}{}^{i} \mathscr{D}_{k,130i+j+t}$.

(8) $\mathscr{D}_{k,514n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,256}{}^{i} \mathscr{D}_{k,258i+j+t}$.

(9) $\mathscr{D}_{k,1026n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,512}{}^{i} \mathscr{D}_{k,514i+j+t}$.

(10) $\mathscr{D}_{k,2050n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,1024}{}^{i} \mathscr{D}_{k,1026i+j+t}$.

In general, for $r,n,t \geq 1$, we have

$$\mathscr{D}_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j(-1)^{j+s} \mathscr{L}_{k,2^{r+1}}{}^{i} \mathscr{D}_{k,(2^{r+1}+2)i+j+t}.$$

Theorem 2.8. *For $n,t \geq 1$ and $\mathscr{D}_{k,n} = \mathscr{F}_{k,n}$ or $\mathscr{L}_{k,n}$, we have*

(1) $\mathscr{D}_{k,4n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,2}{}^{-n} \mathscr{D}_{k,6i+j+t}$.

(2) $\mathscr{D}_{k,6n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,4}{}^{-n} \mathscr{D}_{k,10i+j+t}$.

(3) $\mathscr{D}_{k,10n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,8}{}^{-n} \mathscr{D}_{k,18i+j+t}$.

(4) $\mathscr{D}_{k,18n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,16}{}^{-n} \mathscr{D}_{k,34i+j+t}$.

(5) $\mathscr{D}_{k,34n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,32}{}^{-n} \mathscr{D}_{k,66i+j+t}$.

(6) $\mathscr{D}_{k,66n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,64}{}^{-n} \mathscr{D}_{k,130i+j+t}$.

(7) $\mathscr{D}_{k,130n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,128}{}^{-n} \mathscr{D}_{k,258i+j+t}$.

(8) $\mathscr{D}_{k,258n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,256}{}^{-n} \mathscr{D}_{k,514i+j+t}$.

(9) $\mathscr{D}_{k,514n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,512}{}^{-n} \mathscr{D}_{k,1026i+j+t}$.

(10) $\mathscr{D}_{k,1026n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,1024}{}^{-n} \mathscr{D}_{k,2050i+j+t}$.

In general, for $r,n,t \geq 1$, we have

$$\mathscr{D}_{k,(2^{r+1}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j \mathscr{L}_{k,2^{r+1}}{}^{-n} \mathscr{D}_{k,(2^{r+1}+2)i+j+t}.$$

Lemma 2.9. *Let $u = r_1$ or r_2, then for $l_n = \sum_{i=1}^{n} \mathscr{L}_{k,2^i}$ and $n,t \geq 1$, we have*

(1) $1+u^4 = l_1 u^2$.

(2) $1+u^8 = \frac{l_2}{l_1} u^4 = l_2 u^2 - \frac{l_2}{l_1}$.

(3) $1+u^{16} = \frac{l_3}{l_2} u^8 = \frac{l_3}{l_1} u^4 - \frac{l_3}{l_2} = l_3 u^2 - \frac{l_3}{l_1} - \frac{l_3}{l_2}$.

(4) $1+u^{32} = \frac{l_4}{l_3} u^{16} = \frac{l_4}{l_2} u^8 - \frac{l_4}{l_3} = \frac{l_4}{l_1} u^4 - l_4 \left[\frac{1}{l_2} + \frac{1}{l_3}\right] = l_4 u^2 - l_4 \left[\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3}\right]$.

(5) $1+u^{64} = \frac{l_5}{l_4} u^{32} = \frac{l_5}{l_3} u^{16} - \frac{l_5}{l_4} = \frac{l_5}{l_2} u^8 - l_5 \left[\frac{1}{l_3} + \frac{1}{l_4}\right] = \frac{l_5}{l_1} u^4 - l_5 \left[\frac{1}{l_2} + \frac{1}{l_3} + \frac{1}{l_4}\right]$
$= l_5 u^2 - l_5 \left[\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} + \frac{1}{l_4}\right]$.

In general, we have

$$1+u^{2^n} = \begin{cases} \frac{l_{n-1}}{l_{n-2}} u^{2^{n-1}}; \\ \frac{l_{n-1}}{l_{n-t-1}} u^{2^{n-t}} - l_{n-1} \sum_{i=2}^{t} \frac{1}{l_{n-i}}, \quad \text{If } t = 2,3,4,\ldots,n-2\,; \\ l_{n-1} u^2 - l_{n-1} \sum_{i=2}^{n-1} \frac{1}{l_{n-i}}. \end{cases}$$

Theorem 2.10. *For $l_n = \sum_{i=1}^{n} \mathscr{L}_{k,2^i}$, $n,t \geq 1$ and $\mathscr{D}_{k,n} = \mathscr{F}_{k,n}$ or $\mathscr{L}_{k,n}$, we have*

(1) $\mathscr{D}_{k,t+4} = l_1 \mathscr{D}_{k,t+2} - \mathscr{D}_{k,t}$.

(2) $\mathscr{D}_{k,t+8} = \frac{l_2}{l_1} \mathscr{D}_{k,t+4} - \mathscr{D}_{k,t} = l_2 \mathscr{D}_{k,t+2} - (1+\frac{l_2}{l_1}) \mathscr{D}_{k,t}$.

(3) $\mathscr{D}_{k,t+16} = \frac{l_3}{l_2} \mathscr{D}_{k,t+8} - \mathscr{D}_{k,t}$,
$= \frac{l_3}{l_1} \mathscr{D}_{k,t+4} - (1+\frac{l_3}{l_2}) \mathscr{D}_{k,t}$,
$= l_3 \mathscr{D}_{k,t+2} - (1+\frac{l_3}{l_1}+\frac{l_3}{l_2}) \mathscr{D}_{k,t}$.

(4) $\mathscr{D}_{k,t+32} = \frac{l_4}{l_3} \mathscr{D}_{k,t+16} - \mathscr{D}_{k,t}$,
$= \frac{l_4}{l_2} \mathscr{D}_{k,t+8} - (1+\frac{l_4}{l_3}) \mathscr{D}_{k,t}$,
$= \frac{l_4}{l_1} \mathscr{D}_{k,t+4} - (1+\frac{l_4}{l_2}+\frac{l_4}{l_3}) \mathscr{D}_{k,t}$,
$= l_4 \mathscr{D}_{k,t+2} - (1+\frac{l_4}{l_1}+\frac{l_4}{l_2}+\frac{l_4}{l_3}) \mathscr{D}_{k,t}$.

(5) $\mathscr{D}_{k,t+64} = \frac{l_5}{l_4} \mathscr{D}_{k,t+32} - \mathscr{D}_{k,t}$,
$= \frac{l_5}{l_3} \mathscr{D}_{k,t+16} - (1+\frac{l_5}{l_4}) \mathscr{D}_{k,t}$,
$= \frac{l_5}{l_2} \mathscr{D}_{k,t+8} - (1+\frac{l_5}{l_3}+\frac{l_5}{l_4}) \mathscr{D}_{k,t}$,

$$= \frac{l_5}{l_1}\mathscr{M}_{k,t+4} - (1+\frac{l_5}{l_2}+\frac{l_5}{l_3}+\frac{l_5}{l_4})\mathscr{D}_{k,t},$$
$$= l_5\mathscr{D}_{k,t+2} - (1+\frac{l_5}{l_1}+\frac{l_5}{l_2}+\frac{l_5}{l_3}+\frac{l_5}{l_4})\mathscr{D}_{k,t}.$$

In general, we have

$$\mathscr{D}_{k,t+2^n} = \begin{cases} \frac{l_{n-1}}{l_{n-2}}\mathscr{D}_{k,t+2^{n-1}} - \mathscr{D}_{k,t}; \\ \frac{l_{n-1}}{l_{n-t-1}}\mathscr{D}_{k,t+2^{n-s}} - l_{n-1}\sum_{i=2}^{s}(1+\frac{1}{l_{n-i}})\mathscr{D}_{k,t}, \quad \text{If } s = 2,3,4,\ldots,n-2\,; \\ l_{n-1}\mathscr{D}_{k,t+2} - l_{n-1}\sum_{i=2}^{n-1}(\frac{1}{l_{n-i}}+1)\mathscr{D}_{k,t}. \end{cases}$$

Theorem 2.11. *For $l_n = \sum_{i=1}^{n}\mathscr{L}_{k,2^i}$, $n,t \geq 1$ and $\mathscr{D}_{k,n} = \mathscr{F}_{k,n}$ or $\mathscr{L}_{k,n}$, we have*

(1) $\mathscr{D}_{k,4n+t} = \sum_{i+j=n}\binom{n}{i} l_1^i(-1)^j\mathscr{D}_{k,2i+t}.$

(2) $\mathscr{D}_{k,8n+t} = \sum_{i+j=n}\binom{n}{i}(\frac{l_2}{l_1})^i(-1)^j\mathscr{D}_{k,4i+t},$

$= \sum_{i+j=n}\binom{n}{i} l_2^i(-1)^j(\frac{l_1+l_2}{l_1})\mathscr{D}_{k,2i+t}.$

(3) $\mathscr{D}_{k,16n+t} = \sum_{i+j=n}\binom{n}{i}(\frac{l_3}{l_2})^i(-1)^j\mathscr{D}_{k,8i+t},$

$= \sum_{i+j=n}\binom{n}{i}(\frac{l_3}{l_1})^i(-1)^j(1+\frac{l_3}{l_2})\mathscr{D}_{k,4i+t},$

$= \sum_{i+j=n}\binom{n}{i} {l_3}^i(-1)^j(1+\frac{l_3}{l_1}+\frac{l_3}{l_2})\mathscr{D}_{k,2i+t}.$

(4) $\mathscr{D}_{k,32n+t} = \sum_{i+j=n}\binom{n}{i}(\frac{l_4}{l_3})^i(-1)^j\mathscr{D}_{k,16i+t},$

$= \sum_{i+j=n}\binom{n}{i}(\frac{l_4}{l_2})^i(-1)^j(1+\frac{l_4}{l_3})\mathscr{D}_{k,8i+t},$

$= \sum_{i+j=n}\binom{n}{i}(\frac{l_4}{l_1})^i(-1)^j(1+\frac{l_4}{l_2}+\frac{l_4}{l_3})\mathscr{D}_{k,4i+t},$

$= \sum_{i+j=n}\binom{n}{i} {l_4}^i(-1)^j(1+\frac{l_4}{l_1}+\frac{l_4}{l_2}+\frac{l_4}{l_3})\mathscr{D}_{k,2i+t}.$

(5) $\mathscr{D}_{k,64n+t} = \sum_{i+j=n}\binom{n}{i}(\frac{l_5}{l_4})^i(-1)^j\mathscr{D}_{k,32i+t},$

$= \sum_{i+j=n}\binom{n}{i}(\frac{l_5}{l_3})^i(-1)^j(1+\frac{l_5}{l_4})\mathscr{D}_{k,16i+t},$

$= \sum_{i+j=n}\binom{n}{i}(\frac{l_5}{l_2})^i(-1)^j(1+\frac{l_5}{l_3}+\frac{l_5}{l_4})\mathscr{D}_{k,8i+t},$

$= \sum_{i+j=n}\binom{n}{i}(\frac{l_5}{l_1})^i(-1)^j(1+\frac{l_5}{l_2}+\frac{l_5}{l_3}+\frac{l_5}{l_4})\mathscr{D}_{k,4i+t},$

$= \sum_{i+j=n}\binom{n}{i} {l_5}^i(-1)^j(1+\frac{l_5}{l_1}+\frac{l_5}{l_2}+\frac{l_5}{l_3}+\frac{l_5}{l_4})\mathscr{D}_{k,2i+t}.$

In general, we have

$$\mathscr{D}_{k,2^r n+t} = \begin{cases} \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-2}})^i (-1)^j \mathscr{D}_{k,2^{r-1}i+t}; \\ \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-s-1}})^i (-1)^j (\sum_{h=2}^{s} (1+\frac{l_{r-1}}{l_{r-h}})^j \mathscr{D}_{k,2^{n-s}i+t}, \\ \text{If } s = 2,3,4,\ldots,n-2\ ; \\ \sum_{i+j=n} \binom{n}{i} (l_{r-1})^i (-1)^j (\sum_{h=2}^{s} (1+\frac{l_{r-1}}{l_{r-h}})^j \mathscr{D}_{k,2i+t}. \end{cases}$$

Lemma 2.12. *For $t \geq 1$, we have*

(1) $r_1^2 = r_1\sqrt{\delta} - 1$,
$r_2^2 = -r_2\sqrt{\delta} - 1$.
(2) $r_1^4 = (k^2+2)r_1\sqrt{\delta} - (k^2+3)$,
$r_2^4 = -(k^2+2)r_2\sqrt{\delta} - (k^2+3)$.
(3) $r_1^6 = (k^2+1)(k^2+3)r_1\sqrt{\delta} - (k^4+5k^2+5)$,
$r_2^6 = -(k^2+1)(k^2+3)r_2\sqrt{\delta} - (k^4+5k^2+5)$.
(4) $r_1^8 = (k^2+2)(k^4+4k^2+2)r_1\sqrt{\delta} - (k^6+7k^4+14k^2+7)$,
$r_2^8 = -(k^2+2)(k^4+4k^2+2)r_2\sqrt{\delta} - (k^6+7k^4+14k^2+7)$.
(5) $r_1^{10} = (k^4+3k^2+1)(k^4+5k^2+5)r_1\sqrt{\delta} - (k^2+3)(k^6+6k^4+9k^2+3)$,
$r_2^{10} = -(k^4+3k^2+1)(k^4+5k^2+5)r_2\sqrt{\delta} - (k^2+3)(k^6+6k^4+9k^2+3)$.

In general, we have

$$r_1^{2t} = \frac{\mathscr{F}_{k,2t}}{k} r_1\sqrt{\delta} - \frac{\mathscr{L}_{k,2t-1}}{k},$$
$$r_2^{2t} = -\frac{\mathscr{F}_{k,2t}}{k} r_2\sqrt{\delta} - \frac{\mathscr{L}_{k,2t-1}}{k}.$$

Lemma 2.13. *For $t \geq 1$, we have*

(1) $r_1^3 = (k^2+3)r_1 - \sqrt{\delta}$,
$r_2^3 = (k^2+3)r_2 + \sqrt{\delta}$.
(2) $r_1^5 = (k^4+5k^2+5)r_1 - (k^2+2)\sqrt{\delta}$,
$r_2^5 = (k^4+5k^2+5)r_2 + (k^2+2)\sqrt{\delta}$.
(3) $r_1^7 = (k^6+7k^4+14k^2+7)r_1 - (k^2+1)(k^2+3)\sqrt{\delta}$,
$r_2^7 = (k^6+7k^4+14k^2+7)r_2 + (k^2+1)(k^2+3)\sqrt{\delta}$.
(4) $r_1^9 = (k^2+3)(k^6+6k^4+9k^2+3)r_1 - (k^2+2)(k^4+4k^2+2)\sqrt{\delta}$,
$r_2^9 = (k^2+3)(k^6+6k^4+9k^2+3)r_2 + (k^2+2)(k^4+4k^2+2)\sqrt{\delta}$,.
(5) $r_1^{11} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)r_1 + (k^4+3k^2+1)(k^4+5k^2+5)\sqrt{\delta}$,
$r_2^{11} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)r_2 - (k^4+3k^2+1)(k^4+5k^2+5)\sqrt{\delta}$.

In general, we have

$$r_1^{2t+1} = \frac{\mathscr{L}_{k,2t+1}}{k} r_1 - \frac{\mathscr{F}_{k,2t}}{k}\sqrt{\delta},$$
$$r_2^{2t+1} = \frac{\mathscr{L}_{k,2t+1}}{k} r_2 + \frac{\mathscr{F}_{k,2t}}{k}\sqrt{\delta}.$$

Theorem 2.14. *For $s,t \geq 1$, we have*

(1) $\mathscr{F}_{k,s+2} + \mathscr{F}_{k,s} = \mathscr{L}_{k,s+1}$,
$\mathscr{L}_{k,s+2} + \mathscr{L}_{k,s} = \delta \mathscr{F}_{k,s+1}$.
(2) $\mathscr{F}_{k,s+4} + (k^2+3)\mathscr{F}_{k,s} = (k^2+2)\mathscr{L}_{k,s+1}$,
$\mathscr{L}_{k,s+4} + (k^2+3)\mathscr{L}_{k,s} = (k^2+2)\delta \mathscr{F}_{k,s+1}$.
(3) $\mathscr{F}_{k,s+6} + (k^4+5k^2+5)\mathscr{F}_{k,s} = (k^2+1)(k^2+3)\mathscr{L}_{k,s+1}$,
$\mathscr{L}_{k,s+6} + (k^4+5k^2+5)\mathscr{L}_{k,s} = (k^2+1)(k^2+3)\delta \mathscr{F}_{k,s+1}$.
(4) $\mathscr{F}_{k,s+8} + (k^6+7k^4+14k^2+7)\mathscr{F}_{k,s} = (k^2+2)(k^4+4k^2+2)\mathscr{L}_{k,s+1}$,
$\mathscr{L}_{k,s+8} + (k^6+7k^4+14k^2+7)\mathscr{L}_{k,s} = (k^2+2)(k^4+4k^2+2)\delta \mathscr{F}_{k,s+1}$.
(5) $\mathscr{F}_{k,s+10} + (k^2+3)(k^6+6k^4+9k^2+3)\mathscr{F}_{k,s} = (k^4+3k^2+1)(k^4+5k^2+5)\mathscr{L}_{k,s+1}$,
$\mathscr{L}_{k,s+10} + (k^2+3)(k^6+6k^4+9k^2+3)\mathscr{L}_{k,s} = (k^4+3k^2+1)(k^4+5k^2+5)\delta \mathscr{F}_{k,s+1}$.

In general, we have

$$\mathscr{F}_{k,s+2t} + \frac{\mathscr{L}_{k,2t-1}}{k}\mathscr{F}_{k,s} = \frac{\mathscr{F}_{k,2t}}{k}\mathscr{L}_{k,s+1}, \tag{2.1}$$

$$\mathscr{L}_{k,s+10} + \frac{\mathscr{L}_{k,2t-1}}{k}\mathscr{L}_{k,s} = \frac{\mathscr{F}_{k,2t}}{k}\delta \mathscr{F}_{k,s+1}. \tag{2.2}$$

Remark 2.15. Using $\mathscr{L}_{k,2t-1} - \mathscr{F}_{k,2t} = \mathscr{F}_{k,2t-2}$ in (2.1), we get

$$\mathscr{F}_{k,s+2t} - \frac{\mathscr{F}_{k,2t}}{k}\mathscr{F}_{k,s+2} + \frac{\mathscr{F}_{k,2t-2}}{k}\mathscr{F}_{k,s} = 0,$$

$$\mathscr{L}_{k,s+2t} - \frac{\mathscr{F}_{k,2t}}{k}\mathscr{L}_{k,s+2} + \frac{\mathscr{F}_{k,2t-2}}{k}\mathscr{L}_{k,s} = 0.$$

Theorem 2.16. *For $s,t \geq 1$, we have*

(1) $\mathscr{F}_{k,s+3} + \mathscr{L}_{k,s} = (k^2+3)\mathscr{F}_{k,s+1}$,
$\mathscr{L}_{k,s+3} + \delta \mathscr{F}_{k,s} = (k^2+3)\mathscr{L}_{k,s+1}$.
(2) $\mathscr{F}_{k,s+5} + (k^2+2)\mathscr{L}_{k,s} = (k^4+5k^2+5)\mathscr{F}_{k,s+1}$,
$\mathscr{L}_{k,s+5} + \delta(k^2+2)\mathscr{F}_{k,s} = (k^4+5k^2+5)\mathscr{L}_{k,s+1}$.
(3) $\mathscr{F}_{k,s+7} + (k^2+1)(k^2+3)\mathscr{L}_{k,s} = (k^6+7k^4+14k^2+7)\mathscr{F}_{k,s+1}$,
$\mathscr{L}_{k,s+7} + \delta(k^2+1)(k^2+3)\mathscr{F}_{k,s} = (k^6+7k^4+14k^2+7)\mathscr{L}_{k,s+1}$.
(4) $\mathscr{F}_{k,s+9} + (k^2+2)(k^4+4k^2+2)\mathscr{L}_{k,s} = (k^2+3)(k^6+6k^4+9k^2+3)\mathscr{F}_{k,s+1}$,
$\mathscr{L}_{k,s+9} + \delta(k^2+2)(k^4+4k^2+2)\mathscr{F}_{k,s} = (k^2+3)(k^6+6k^4+9k^2+3)\mathscr{L}_{k,s+1}$.
(5) $\mathscr{F}_{k,s+11} + (k^4+3k^2+1)(k^4+5k^2+5)\mathscr{L}_{k,s} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)\mathscr{F}_{k,s+1}$,
$\mathscr{L}_{k,s+11} + \delta(k^4+3k^2+1)(k^4+5k^2+5)\mathscr{F}_{k,s} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)\mathscr{L}_{k,s+1}$.

In general, we have

$$\mathscr{F}_{k,s+2t+1} + \frac{\mathscr{F}_{k,2t}}{k}\mathscr{L}_{k,s} = \frac{\mathscr{L}_{k,2t+1}}{k}\mathscr{F}_{k,s+1}, \tag{2.3}$$

$$\mathscr{L}_{k,s+2t+1} + \delta\frac{\mathscr{F}_{k,2t}}{k}\mathscr{F}_{k,s} = \frac{\mathscr{L}_{k,2t+1}}{k}\mathscr{L}_{k,s+1}. \tag{2.4}$$

Remark 2.17. Using $(k^2+3)\mathscr{F}_{k,2t} - \mathscr{L}_{k,2t-1} = \mathscr{F}_{k,2t-2}$ in (2.3), we obtain

$$\mathscr{F}_{k,s+2t+1} - \frac{\mathscr{L}_{k,2t+1}}{k(k^2+3)}\mathscr{F}_{k,s+3} + \frac{\mathscr{F}_{k,2t-2}}{k(k^2+3)}\mathscr{L}_{k,s} = 0,$$

$$\mathscr{L}_{k,s+2t+1} - \frac{\mathscr{L}_{k,2t+1}}{k(k^2+3)}\mathscr{L}_{k,s+3} + \frac{\mathscr{F}_{k,2t-2}}{k(k^2+3)}\delta \mathscr{F}_{k,s} = 0.$$

Theorem 2.18. *For $n,s,t \geq 1$, we have*

(1) $$\sum_{i=0}^{n}\binom{n}{i}\mathscr{F}_{k,2i+s} = \begin{cases} \delta^{\frac{n}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is even;} \\ \delta^{\frac{n-1}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}\mathscr{L}_{k,2i+s} = \begin{cases} \delta^{\frac{n}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is even;} \\ \delta^{\frac{n+1}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is odd.} \end{cases}$$

(2) $$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathscr{F}_{k,4i+s} = \begin{cases} (k^2+2)^n\delta^{\frac{n}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+2)^n\delta^{\frac{n-1}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathscr{L}_{k,4i+s} = \begin{cases} (k^2+2)^n\delta^{\frac{n}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+2)^n\delta^{\frac{n+1}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is odd.} \end{cases}$$

(3) $$\sum_{i=0}^{n}\binom{n}{i}(k^4+5k^2+5)^{(n-i)}\mathscr{F}_{k,6i+s} = \begin{cases} (k^2+1)^n(k^2+3)^n\delta^{\frac{n}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+1)^n(k^2+3)^n\delta^{\frac{n-1}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^4+5k^2+5)^{(n-i)}\mathscr{L}_{k,6i+s} = \begin{cases} (k^2+1)^n(k^2+3)^n\delta^{\frac{n}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+1)^n(k^2+3)^n\delta^{\frac{n+1}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is odd.} \end{cases}$$

(4) $$\sum_{i=0}^{n}\binom{n}{i}(k^6+7k^4+14k^2+7)^{(n-i)}\mathscr{F}_{k,8i+s} = \begin{cases} (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n-1}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^6+7k^4+14k^2+7)^{(n-i)}\mathscr{L}_{k,8i+s} = \begin{cases} (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n+1}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is odd.} \end{cases}$$

(5) $$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(k^6+6k^4+9k^2+3)^{(n-i)}\mathscr{F}_{k,10i+s} = \begin{cases} (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n-1}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(k^6+6k^4+9k^2+3)^{(n-i)}\mathscr{L}_{k,10i+s} = \begin{cases} (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n+1}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is odd.} \end{cases}$$

In general, for $n,s,t \geq 1$, we have

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(\mathscr{L}_{k,2t-1})^{(n-i)}\mathscr{F}_{k,2ti+s} = \begin{cases} k^{-n}(\mathscr{F}_{k,2t})^n\delta^{\frac{n}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is even;} \\ k^{-n}(\mathscr{F}_{k,2t})^n\delta^{\frac{n-1}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(\mathscr{L}_{k,2t-1})^{(n-i)}\mathscr{L}_{k,2ti+s} = \begin{cases} k^{-n}(\mathscr{F}_{k,2t})^n\delta^{\frac{n}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is even;} \\ k^{-n}(\mathscr{F}_{k,2t})^n\delta^{\frac{n+1}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 2.19. *For $n,s,t \geq 1$, we have*

(1) $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i\mathscr{F}_{k,2(n-i)+n} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 2\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i\mathscr{L}_{k,2(n-i)+n} = \begin{cases} 2\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(2) $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^4+5k^2+5)^i\mathscr{F}_{k,4(n-i)+n} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 2(k^2+2)^n\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^4+5k^2+5)^i\mathscr{N}_{k,4(n-i)+n} = \begin{cases} 2(k^2+2)^n\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(3) $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^6+7k^4+14k^2+7)^i\mathscr{F}_{k,6(n-i)+n}$$
$$= \begin{cases} 0, & \text{if } n \text{ is even;} \\ 2(k^2+1)^n(k^2+3)^n\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^6+7k^4+14k^2+7)^i\mathscr{L}_{k,6(n-i)+n}$$
$$= \begin{cases} 2(k^2+1)^n(k^2+3)^n\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(4) $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i(k^6+6k^4+9k^2+3)^i\mathscr{F}_{k,8(n-i)+n}$$
$$= \begin{cases} 0, & \text{if } n \text{ is even;} \\ 2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i(k^6+6k^4+9k^2+3)^i\mathscr{L}_{k,8(n-i)+n}$$
$$= \begin{cases} 2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(5) $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^{10}+11k^8+44k^6+44k^4+55k^2+11)^i\mathscr{F}_{k,10(n-i)+n}$$
$$= \begin{cases} 0, & \text{if } n \text{ is even;} \\ 2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^{10}+11k^8+44k^6+44k^4+55k^2+11)^i\mathscr{L}_{k,10(n-i)+n}$$
$$= \begin{cases} 2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

In general, for $n,s,t \geq 1$, we have
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^i\mathscr{F}_{k,2t(n-i)+n} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 2(k)^{-n}(\mathscr{F}_{k,2t})^n\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^i\mathscr{L}_{k,2t(n-i)+n} = \begin{cases} 2(k)^{-n}(\mathscr{F}_{k,2t})^n\delta^{\frac{n-1}{2}}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

In next section, we prove some elementary and binomial properties of k-Fibonacci and k-Lucas sequences.

3. The Proofs of the Main Results

Proof of Lemma(2.1): We prove only (a), (c) and (d) since the proofs of (b) and (e) are similar.

Proof of (a): Since r_1 and r_2 are roots of $r^2 - kr - 1 = 0$, then

$$r_1^2 = kr_1 + 1, \tag{3.1}$$

$$r_2^2 = kr_2 + 1. \tag{3.2}$$

This completes the proof of (a).

Proof of (c): From (b), we have

$$\begin{aligned} u^{2n} &= \mathscr{F}_{k,n}u^{n+1} + u^n\mathscr{F}_{k,n-1} \\ &= \mathscr{F}_{k,n}(u\mathscr{F}_{k,n+1} + \mathscr{F}_{k,n}) + u^n\mathscr{F}_{k,n-1} \\ &= u\mathscr{F}_{k,n}\mathscr{F}_{k,n+1} + \mathscr{F}_{k,n-1}u^n + \mathscr{F}_{k,n}^2 \\ &= (u^n - \mathscr{F}_{k,n-1})\mathscr{F}_{k,n+1} + \mathscr{F}_{k,n-1}u^n + \mathscr{F}_{k,n}^2 \\ &= u^n(\mathscr{F}_{k,n+1} + \mathscr{F}_{k,n-1}) + \mathscr{F}_{k,n}^2 - \mathscr{F}_{k,n}F_{k,n-1}. \end{aligned}$$

Using $\mathscr{F}_{k,n-1}\mathscr{F}_{k,n+1} - \mathscr{F}_{k,n}^2 = (-1)^n$ and $\mathscr{F}_{k,n+1} + \mathscr{F}_{k,n-1} = \mathscr{L}_{k,n}$, we obtain

$$u^{2n} = \mathscr{L}_{k,n}u^n - (-1)^n.$$

This completes the proof of (c).

Proof of (d): If $u = r_1$, then we have

$$\begin{aligned} \mathscr{F}_{k,tn}r_1^n - (-1)^n\mathscr{F}_{k,(t-1)n} &= (\frac{r_1^{tn} - r_2^{tn}}{r_1 - r_2})r_1^n - (r_1r_2)^n(\frac{r_1^{(t-1)n} - r_2^{(t-1)n}}{r_1 - r_2}) \\ &= (\frac{r_1^n - r_2^n}{r_1 - r_2})r_1^{tn} \\ &= \mathscr{F}_{k,n}r_1^{tn}. \end{aligned}$$

This completes the proof of (d).

The proofs of Theorems (2.5), (2.10) are similar. Hence, we prove only Theorem (2.2).

Proof of Theorem(2.2): We prove only (a), since the proofs of (b), (c) and (d) are similar.

Proof of (1): From 2.1(b), we have

$$r_1^n = \mathscr{F}_{k,n}r_1 + \mathscr{F}_{k,n-1}, \tag{3.3}$$

$$r_2^n = \mathscr{F}_{k,n}r_2 + \mathscr{F}_{k,n-1}. \tag{3.4}$$

Multiplying (3.3) by r_1^t, (3.4) by r_2^t and subtracting, we obtain

$$\frac{r_1^{n+t}-r_2^{n+t}}{r_1-r_2}=\mathscr{F}_{k,n}(\frac{r_1^{t+1}-r_2^{t+1}}{r_1-r_2})+\mathscr{F}_{k,n-1}(\frac{r_1^{t}-r_2^{t}}{r_1-r_2}).$$

Hence, it gives that

$$\mathscr{F}_{k,n+t}=\mathscr{F}_{k,n}\mathscr{F}_{k,t+1}+\mathscr{F}_{k,n-1}\mathscr{F}_{k,t}.$$

This completes the proof of (a).

The proofs of Theorems (2.6)-(2.8) and (2.11) are similar. Hence, we prove only Theorem (2.3).

Proof of Theorem(2.3): We prove only (3), since the proofs of (1), (2) and (4)-(8) are similar.

*Proof of (3):*From 2.1(b), we have

(3.5) $$r_1^r=\mathscr{F}_{k,r}r_1+\mathscr{F}_{k,r-1},$$

(3.6) $$r_2^r=\mathscr{F}_{k,r}r_2+\mathscr{F}_{k,r-1}.$$

Now, by the binomial theorem, we have

(3.7) $$r_1^{rn}=\sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}r_1^{i},$$

(3.8) $$r_2^{rn}=\sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}r_2^{i}.$$

Now, by subtracting (3.7) from (3.8), we obtain

$$\frac{r_1^{rn+t}-r_2^{rn+t}}{r_1-r_2}=\sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}\mathscr{F}_{k,r-1}^{n-i}(\frac{r_1^{i+t}-r_2^{i+t}}{r_1-r_2}).$$

Hence, it gives that

$$\mathscr{F}_{k,rn+t}=\sum_{i=0}^{n}\binom{n}{i}\mathscr{F}_{k,r}^{i}\mathscr{F}_{k,r-1}^{n-i}\mathscr{F}_{k,i+t}.$$

Now, by adding (3.7) and (3.8), we get

$$r_1^{rn+t}+r_2^{rn+t}=\sum_{i=0}^{n}\binom{n}{i}\mathscr{F}_{k,r}^{i}\mathscr{F}_{k,r-1}^{n-i}(r_1^{i+t}+r_2^{i+t}).$$

Hence, it gives that

$$\mathscr{L}_{k,rn+t}=\sum_{i=0}^{n}\binom{n}{i}\mathscr{F}_{k,r}^{i}\mathscr{F}_{k,r-1}^{n-i}\mathscr{L}_{k,i+t}.$$

This completes the proof of (3).

Proof of Lemma(2.4): We prove only (1) and (2) since the proofs of (3)-(11) are similar.

Proof of (1): Using (3.1) and (3.2), we have

$$\begin{aligned} u^3 &= u^2u \\ &= (ku+1)u \\ &= ku^2+u \\ &= k(ku+1)+u \\ &= k^2u+k+u \\ &= k+(k^2+1)u. \end{aligned}$$

This completes the proof of (1).

Proof of (2): Using (3.1) and (3.2), we have

$$\begin{aligned} 1+ku++u^6 &= u^2+u^6 \\ &= u^2+u^4(ku+1) \\ &= u^2+ku^5+u^4 \\ &= u^2+ku^3(ku+1)+u^4 \\ &= u^2+k^2u^4+ku^3+u^4 \\ &= (k^2+1)u^4+ku^3+u^2 \\ &= (k^2+1)u^4+u^2(ku+1) \\ &= (k^2+1)u^4+u^4 \\ &= (k^2+2)u^4 \\ &= F_{k,2}u^4. \end{aligned}$$

This completes the proof of (2). The proofs of lemma (2.13) are similar. Hence, we prove only Lemma (2.12).

Proof of Lemma(2.12): We prove only (1) and (2) since the proofs of (3) - (5) are similar.

Proof of (1): Using $r_1 - r_2 = \sqrt{\delta}$, we have

$$\begin{aligned} r_1\sqrt{\delta}-1 &= r_1(r_1-r_2)-1 \\ &= r_1^2-r_1r_2-1 \\ &= r_1^2+1-1 \\ &= r_1^2. \end{aligned}$$

This completes the proof of (1).

Proof of (2): Using (3.1) and (3.2), we have

$$\begin{aligned}(k^2+2)r_1\sqrt{\delta}-(k^2+3) &= (k^2+2)r_1(r_1-r_2)-(k^2+3)\\ &= (k^2+2)(r_1^2-r_1r_2)-(k^2+3)\\ &= (k^2+2)(r_1^2+1)-(k^2+3)\\ &= r_1^2(k^2+2)+(k^2+2)-(k^2+3)\\ &= r_1^2k^2+2r_1^2-1\\ &= r_1^2k^2+2(kr_1+1)-1\\ &= r_1^2k^2+2kr_1+1\\ &= r_1^2k^2+kr_1+kr_1+1\\ &= (kr_1+1)(kr_1+1)\\ &= r_1^2r_1^2\\ &= r_1^4.\end{aligned}$$

This completes the proof of (2).

The proofs of Theorems (2.14) and (2.16) are similar. Hence, we prove only Theorem (2.14).

Proof of Theorem(2.14): We prove only (2), since the proofs of (1) and (3)-(5) are similar.

*Proof of (2):*From 2.12(2), we have

(3.9) $$r_1^4+(k^2+3)=(k^2+2)r_1\sqrt{\delta},$$

(3.10) $$r_2^4+(k^2+3)=-(k^2+2)r_2\sqrt{\delta}.$$

Multiplying (3.9) by r_1^s, (3.10) by r_2^s and subtracting, we obtain

$$\frac{r_1^{s+4}-r_2^{s+4}}{r_1-r_2}+(k^2+3)\frac{r_1^s-r_2^s}{r_1-r_2}=(k^2+2)(r_1^{s+1}+r_2^{s+1})$$

Hence, it gives that

$$\mathscr{F}_{k,s+4}+(k^2+3)\mathscr{F}_{k,s}=(k^2+2)\mathscr{L}_{k,s+1}.$$

Multiplying (3.9) by r_1^s, (3.10) by r_2^s and adding, we obtain

$$r_1^{s+4}+r_2^{s+4}+(k^2+3)(r_1^s+r_2^s)=(k^2+2)\delta(\frac{r_1^{s+1}-r_2^{s+1}}{r_1-r_2})$$

Hence, it gives that

$$\mathscr{L}_{k,s+4}+(k^2+3)\mathscr{L}_{k,s}=(k^2+2)\delta\mathscr{F}_{k,s+1}.$$

This completes the proof of (3).

The proofs of Theorems (2.18) and (2.19) are similar. Hence, we prove only Theorem (2.18).

Proof of Theorem(2.18): We prove only (2), since the proofs of (1)and (3)-(5) are similar.

*Proof of (2):*From 2.12(2), we have

$$r_1^4 + (k^2+3) = (k^2+2)r_1\sqrt{\delta},$$

$$r_2^4 + (k^2+3) = -(k^2+2)r_2\sqrt{\delta}.$$

Now, by the binomial theorem, we have

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(r_1^{4i+s}) = (k^2+2)^n\delta^{\frac{n}{2}}(r_1^{n+s}), \tag{3.11}$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(r_2^{4i+s}) = (-1)^n(k^2+2)^n\delta^{\frac{n}{2}}(r_2^{n+s}). \tag{3.12}$$

Now, by subtracting (3.11) from (3.12), we obtain

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(\frac{r_1^{4i+s}-r_2^{4i+s}}{r_1-r_2}) = (k^2+2)^n\delta^{\frac{n}{2}}(\frac{r_1^{n+s}-(-1)^n r_2^{n+s}}{r_1-r_2}).$$

Hence, it gives that

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathscr{F}_{k,4i+s} = \begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is even;}\\ (k^2+2)^n\delta^{\frac{n-1}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

Now, by adding (3.11) and (3.12), we get

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(r_1^{4i+s}+r_2^{4i+s}) = (k^2+2)^n\delta^{\frac{n}{2}}(\bar{r}_1 r_1^{n+s}+(-1)^n r_2^{n+s}).$$

Hence, it gives that

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathscr{L}_{k,4i+s} = \begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathscr{L}_{k,n+s}, & \text{if } n \text{ is even;}\\ (k^2+2)^n\delta^{\frac{n+1}{2}}\mathscr{F}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

This completes the proof of (3).

In next section, we investigate certain congruence properties of k-Fibonacci and k-Lucas sequences.

4. Some Congruence Properties of the Generalized k-Lucas Sequence

Theorem 4.1. *For $n,t \geq 1$ and $\mathscr{D}_{k,n} = \mathscr{F}_{k,n}$ or $\mathscr{L}_{k,n}$, we have*

(1) $\mathscr{D}_{k,n+t} - \sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^n\mathscr{D}_{k,6j+t} \equiv 0 \pmod{L_{k,2}}$.

(2) $\mathscr{D}_{k,n+t} - \sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^n\mathscr{D}_{k,10j+t} \equiv 0 \pmod{L_{k,4}}$.

(3) $\mathscr{D}_{k,n+t} - \sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^n\mathscr{D}_{k,18j+t} \equiv 0 \pmod{L_{k,8}}$.

(4) $\mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathscr{D}_{k,34j+t} \equiv 0 \pmod{L_{k,16}}$.

(5) $\mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathscr{D}_{k,66j+t} \equiv 0 \pmod{L_{k,32}}$.

(6) $\mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathscr{D}_{k,130j+t} \equiv 0 \pmod{L_{k,64}}$.

(7) $\mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathscr{D}_{k,258j+t} \equiv 0 \pmod{L_{k,128}}$.

(8) $\mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathscr{D}_{k,514j+t} \equiv 0 \pmod{L_{k,256}}$.

(9) $\mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathscr{D}_{k,1026j+t} \equiv 0 \pmod{L_{k,512}}$.

(10) $\mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathscr{D}_{k,2050j+t} \equiv 0 \pmod{L_{k,1024}}$.

In general, for $r,n,t \geq 1$, we have

$$\mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathscr{D}_{k,(2^{r+2}+2)j+t} \equiv 0 \pmod{L_{k,2^{r+1}}}.$$

Theorem 4.2. *For $n,t \geq 1$ and $\mathscr{D}_{k,n} = \mathscr{F}_{k,n}$ or $\mathscr{L}_{k,n}$, we have*

(1) $\mathscr{D}_{k,6n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,2}}$.

(2) $\mathscr{D}_{k,10n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,4}}$.

(3) $\mathscr{D}_{k,18n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,8}}$.

(4) $\mathscr{D}_{k,34n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,16}}$.

(5) $\mathscr{D}_{k,66n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,32}}$.

(6) $\mathscr{D}_{k,130n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,64}}$.

(7) $\mathscr{D}_{k,258n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,128}}$.

(8) $\mathscr{D}_{k,514n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,256}}$.

(9) $\mathscr{D}_{k,1026n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,514}}$.

(10) $\mathscr{D}_{k,2050n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,1024}}$.

In general, we have

$$\mathscr{D}_{k,(2^{r+2}+2)n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathscr{D}_{k,j+t} \equiv 0 \pmod{L_{k,2^{r+1}}}.$$

The proofs of theorems (4.1) and (4.2) are similar. Hence, we prove only Theorem (4.1).

Proof of Theorem(4.1): We prove only (1), since the proofs of (2)-(10) are similar.

*Proof of (1):*From Theorem (2.6;(1)), For $n,t \geq 1$ and $\mathscr{D}_{k,n} = \mathscr{F}_{k,n}$ or $\mathscr{L}_{k,n}$, we have

$$\mathscr{D}_{k,n+t} = \sum_{i+j+s=n;_{i\neq 0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {\mathscr{L}_{k,2}}^{i} \mathscr{D}_{k,4i+6j+t}$$

$$+ \sum_{i+j+s=n;_{i=0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {\mathscr{L}_{k,2}}^{i} \mathscr{D}_{k,4i+6j+t},$$

$$= \sum_{i+j+s=n;_{i\neq 0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {\mathscr{L}_{k,2}}^{i} \mathscr{D}_{k,4i+6j+t} + \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathscr{D}_{k,6j+t}.$$

$$\mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathscr{D}_{k,6j+t} = \sum_{i+j+s=n;_{i\neq 0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {\mathscr{L}_{k,2}}^{i} \mathscr{D}_{k,4i+6j+t},$$

$$\therefore \mathscr{L}_{k,2} \quad \text{divides} \quad (\mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathscr{D}_{k,6j+t}),$$

$$\therefore \mathscr{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} \mathscr{D}_{k,6j+t} \equiv 0 \pmod{\mathscr{L}_{k,2}}.$$

This completes the proof of (1).

References

[1] A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers", *The Fibonacci Quarterly*, Vol. 03(3), (1965), 61 – 76. http://www.fq.math.ca/Scanned/3-3/horadam.

[2] L. Carlitz, H. H. Ferns. "Some Fibonacci and Lucas Identities", *The Fibonacci Quarterly*, Vol. 08(1), (1970), 61 – 73. http://www.fq.math.ca/Scanned/8-1/carlitz.

[3] Zhizheng Zhang. "Some Identities Involving Generalized Second-Order Integer Sequences", *The Fibonacci Quarterly*, Vol. 35(3), (1997), 265 – 68. http://www.fq.math.ca/Scanned/35-3/ZhizhengZhang.

[4] S. Falcon, A. Plaza. "On k -Fibonacci sequences and polynomials and their derivatives", *Chaos, Solitons and Fractals*, Vol. 39(03), (2009), 1005 – 1019. https://doi.org/10.1016/j.chaos.2007.03.007.

[5] S. Falcon, A. Plaza. "On the k Fibonacci numbers", *Chaos, Solitons and Fractals*, Vol. 32(05), (2007), 1615 – 1624. https://doi.org/10.1016/j.chaos.2006.09.022.

[6] S. Falcon, A. Plaza. "On the k Lucas numbers", *Intl. J. Contemp. Math. Sci.*, Vol. 6, (2011), 1039 – 1050.

[7] C. Bolat. "Properties and applications of k -Fibonacci, k-Lucas numbers", *M. S. Thesis, Selcuk University*, (2008), Konya, Turkey. http://www.m-hikari.com/ijcms-2010/21-24-2010/bolatIJCMS21-24-2010

[8] Paula Catarino, P. Vasco . "Some basic properties and a two-by-two matrix involving the k-Pell numbers", *Int. Journal of Math. Analysis*, Vol. 7(45),(2013), 2209–2215. https://doi.org/10.1016/j.chaos.2015.04.017

[9] S. Falcon, A. Plaza. "On k-Fibonacci numbers of arithmetic indexes", *Applied Mathematics and Computation*, Vol. 208 (1),(2009), 180–185. https://doi.org/10.1016/j.amc.2008.11.031

[10] F. Sergio. "Generalized Fibonacci sequences generated from a k-Fibonacci sequence", *Journal of Mathematics Research*, Vol. 4 (2),(2012), 97–100. http://www.ccsenet.org/journal/index.php/jmr/article/download/14516/10822

[11] A. D. Godase, M. B. Dhakne. "On the properties of k Fibonacci and k Lucas numbers", *International Journal of Advances in Applied Mathematics and Mechanics*, Vol. 02 (2014), 100 – 106. http://www.ijaamm.com/uploads/2/1/4/8/21481830/v2n1p12

[12] Y. Yazlik, N. Yilmaz and N. Taskara. "On the Sums of Powers of k-Fibonacci and k-Lucas Sequences", *Selcuk J. Appl. Math.*, Vol. Special Issue (2012), 47–50.
http://sjam.selcuk.edu.tr/sjam/article/view/327/249

[13] S.Ugyun, and H. Eldogan . "*k*-Jacobsthal and *k*-Jacobsthal Lucas matrix sequences", *International Mathematical Forum*, Vol. 11(3) (2016), 145–154.
http://m-hikari.com/imf-2010/1-4-2010/demirturkIMF1-4-2010

Department of Mathematics,, V. P. College Vaijapur,, Aurangabad(MH),, India.
Email address: vpmaths@datamail.in