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Numerical solution of non linear Burger equation using wavelet approximation combined with finite volume formulation.

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Abstract

The paper proposes a new algorithm which combines the finite volume and wavelet approximation to bring together the salient features of both the approaches for obtaining numerical solution of nonlinear viscous Burger equation under various initial and boundary conditions. The approach is based on approximating the values using wavelet and then it is combined with finite volume formulation. It uses the localization property of wavelet basis. The root mean square error is studied to establish the improvement in the solution as compared to the classical approaches. Plots and tables indicate the significance of the algorithm discussed.

keywords: viscous Burger equation, daubechies bases, finite volume, multi resolution.

AMS Subject Classification: 65T60, 65N08, 65Y20

1 Introduction

The concept of numerical approximation for solution of a partial differential equation, is widely implemented in many applications. Various approaches like wavelet based finite difference [6],[13],[14], finite element[2], finite volume applicaton in shallow water problem [5], their combinations like discontinuous galerikin in shock tube, blast wave problem and Shu Osher problem as in [17], [15] have been used by the researchers.

We consider the classical non linear Burger equation

$$\begin{gathered}\frac{\partial w}{\partial t}+\beta w\frac{\partial w}{\partial x}-\nu\frac{\partial^2 w}{\partial^2 x}=0,\\ a\leq x\leq b,\quad t>0\end{gathered}\tag{1}$$

with initial and boundary conditions as $w(x,0)=f_0(x)$, $a\leq x\leq b$ and $w(a,t)=g_0(t)$ and $w(b,t)=g_1(t)$, $t\in[0,T]$ where $\nu>0$ is a small parameter known as the kinematic viscosity and β is some positive constant. The Burger's equation is the simplest nonlinear model equation for diffusive waves in fluid dynamics. Burger's equation arises in many physical problems including one-dimensional turbulence, sound waves in a viscous medium, shock waves in a viscous medium, waves in fluid filled in viscous elastic tubes, and magneto-hydrodynamic waves in a medium with finite electrical conductivity. The Burger equation was first given by Bateman [9]. The Burger's equation is similar to the one dimensional Navier-Stokes equation without the stress term [20].

Burger equations have been tackled for application with many physical phenomenon such as in understanding turbulence, prediction of features of technologies during turbulent mixing, turbulent convection, turbulent combustion, on the basis of fundamental computational fluid dynamics. Many numerical solutions have been proposed namely B-spline collocation [20], [10], Galerikin finite element approach [1]. We compare our results with [20], [7],[21], [3].

The method proposed here for solving nonlinear Burger equation combines the wavelet based approach and the parabolic method in a novel pattern to bring together the benefits of both the approaches.

The study proved that the error is reduced by introducing the wavelet approximation to finite volume algorithm with various test cases. This is achieved due to the utilization of salient features of conservative property embedded in finite volume approach and localization property of wavelets. The discussion includes description of a function representation by decomposed multi wavelets.

This paper is organized in six sections. Section 1 includes brief literature survey, describes the equation of study, namely the non linear Burger equation and brief introduction of proposed algorithm. Section 2 gives brief overview of the method of discretization namely parabolic method used in this study. Section 3 is about the review of wavelets and multi resolution concept. Section 4 discusses the proposed algorithm. Section 5 gives numerical examples with plots and tables showing the improvement observed in the modified approach along with convergence analysis. Sections 6 gives conclusion and discusses the future direction of research.

2 Discretization

The governing equation (1) in section 1 is discretized using the Parabolic method as in [18]. We rewrite the equation (1) as,

$$w_t + [f(w)]_x = \nu w_{xx}, \quad where \qquad f(w) = \frac{\beta w^2}{2} \tag{2}$$

Now by integrating equation (2) with respect to x between $x_{j-\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$ then following steps in [18] equation (2) is converted into following discreized form ,

$$w_j^{n+1} = w_j^n + k(\nu \frac{w_{j+1}^n - 2w_j^n + w_{j-1}^n}{h^2} + \frac{f[w_{j+\frac{1}{2}}^n] - f[w_{j-\frac{1}{2}}^n]}{h}) \tag{3}$$

where $f[w_{j\pm\frac{1}{2}}^n]$ is the average of $f[w_j^n]$ and $f[w_{j\pm1}^n]$. k and h indicates the time and space step size respectively. Now, the next section discusses the concept of wavelets and multi resolution analysis in brief.

3 Wavelets and Multi resolution

As introduced by S. Mallat and Y. Meyer, a multiresolution of $L^2(R)$ is a specific approximation scheme for finite energy functions. A multiresolution of $L^2[-1,1]$ is a increasing sequence of mutually orthogonal closed linear subspaces as in [8].
Such an infinite nested sequence of subspaces is given by

$$\{V_k^0 \subset V_k^1 \subset .. \subset V_k^n \subset ... \subset L^2[-1,1]\}$$

where, $V_k^n = \{$ f: f is a polynomials of degree $\leq k$ on support of interval$\}$

$$(-1+2^{-n+1}j, -1+2^{-n+1}(j+1)),$$

for $j = 0,1,...,2^n-1$ and vanishes elsewhere.
In particular, this property is valid for a function space
$V_k^0 = \{$ f:f is a polynomial of degree $\leq k$ with support on $[-1,1]\}$ with dimension $k+1$ that can be spanned by a scaling basis ϕ figure 1.
For example if

$$V_k^0 = \{f : f \in P^{k-1}[-1,1]\}$$

$$V_k^1 = \{f : f \in P^{k-1}[-1,0) \cup (0,1]\}$$

and so on are considered with $P^k([-1,1])$ being the space of polynomial of degree k.
From the father basis ϕ , it is possible to span any sub-space V_k^n via dilation and translation [16].

$$\phi_j^n(x) = 2^{(n/2)}\phi(2^n(x+1)-2j-1) \tag{4}$$

with $n = 0,1,2,..N$, $j = 0,1,...2^n-1$ here n is dilation index and j is translation index.
The wavelet sub-spaces W_k^n $(n \geq 0)$ is the orthogonal complement of V_k^n in V_k^{n+1} and they satisfy the conditions:

$$V_k^{n+1} = V_k^n \oplus W_k^n \qquad V_k^n \perp W_k^n$$

Taking Daubechies wavelet ψ as the mother wavelet figure 2, it spans the space W_k^0 and any subspace W_k^n can be spanned by it translation and dilation as,

$$\psi_j^n(x) = 2^{(n/2)}\psi(2^n(x+1)-2j-1) \tag{5}$$

using equation (4) any function can be linearly expressed in single scale decomposition as an orthogonal projection in V_p^n with respect to the bases $\phi_{l,j}^n$ as,

$$P_p^n f(x) = \sum_{j=0}^{2^n-1} \sum_{l=0}^{p} s_{l,j}^n \phi_{l,j}^n \tag{6}$$

p is the order of legendre polynomial used in the generation of scaling space,with $j = 0, 1, ..., 2^n - 1$ as the resolution.

The legendre multiscaling bases $\phi_{l,j}^n$ are obtained by dilation and translation in the interval $[-1, 1]$, followed by $L^2[-1, 1]$ normalization as in [17]. The $\phi_{l,j}$ is the scaling function and the scaling coefficient is given by

$$s_{l,j}^n(i) = \langle f, \phi_{l,j} \rangle$$

which is single scale decomposition. Now using the fact,

$$V_p^N = V_p^0 + W_p^0 + W_p^1 + ... + W_p^{N-1} \tag{7}$$

The multi scale decomposition could be given as

$$P_p^N f(x) = \sum_{l=0}^{p} s_{l,0}^0 \phi_{l,0}^0 + \sum_{n=0}^{N-1} \sum_{j=0}^{2^n-1} \sum_{l=0}^{p} d_{l,j}^n \psi_{l,j}^n(x) \tag{8}$$

with $\psi_{l,j}$ as the wavelet function and its corresponding detail coefficient as $d_{l,j}^n = \langle f, \psi_{l,j} \rangle$. The space spanned by polynomial of degree zero for example is the Haar wavelet family which is utilized as basis as in [5]. In this paper Daubechies wavelet (db2) is utilized, with $p = 1$, $N = 7$ for equation (8). The Daubechies wavelets are not defined in terms of the resulting scaling and wavelet function, in fact they are not possible to write down in closed form. The graph of db2 in figures 1 and 2 is generated using cascade algorithm [12].

4 Algorithm proposed

Algorithm combines the features of discretization and mutiscaling, to obtain a more reliable approach. The underlying idea is as follows,

- The time and space is discretized with $\triangle t$ and $\triangle x$ as per the requirement of the example. Initialization of the viscousity parameter is done.
- The function that represents the solution as per the initial condition is decomposed using multiscale decomposition by equation (8)
- To go to the next time step we use finite volume frame work equation (3) in section 2, and wavelet decomposition is performed as equation (8).

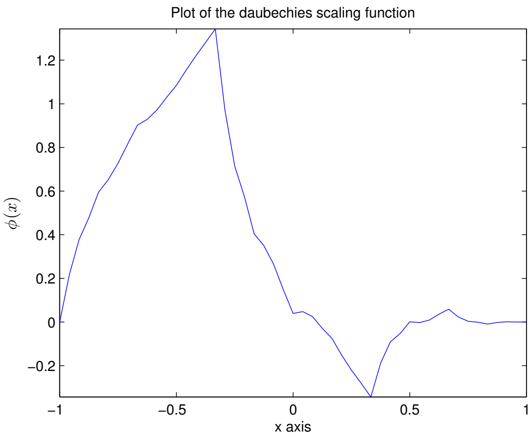

Figure 1: Example of Daubechies scaling function db2

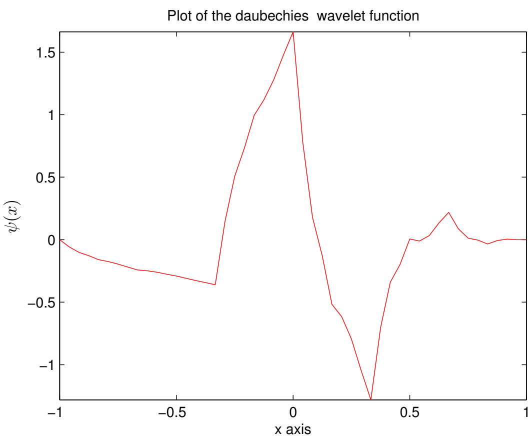

Figure 2: Example of Daubechies wavelet function db2

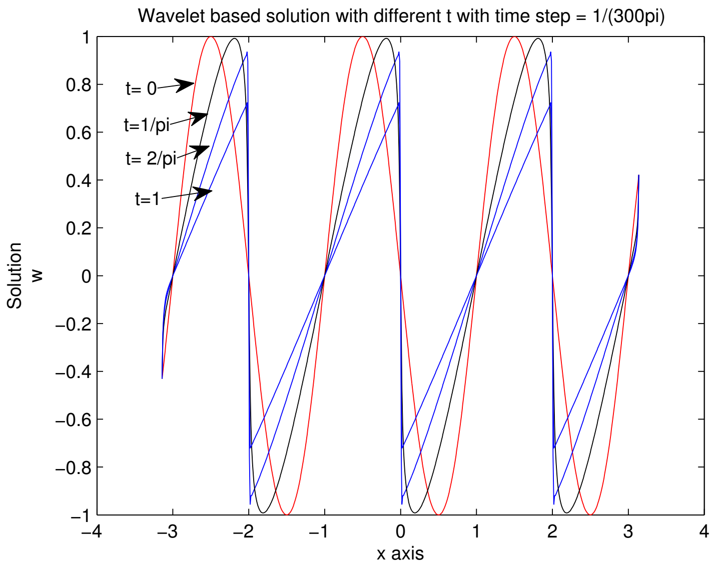

Figure 3: Wavelet based solution plot for the example 1

- The above step is repeated upto time T.
- Solution at all time step is represented in terms of a matrix..

5 Numerical experiments and discussion

In order to justify the implementation and adaptability of the algorithm three examples are discussed.

5.1 Example 1

Consider the equation

$$\frac{\partial w}{\partial t} + \beta w \frac{\partial w}{\partial x} - \nu \frac{\partial^2 w}{\partial^2 x} = 0, \qquad a \leq x \leq b, \quad t > 0$$

taking $\beta = 1$ with initial condition $w(x,0) = -\sin \pi x$, $\nu = \frac{10^{-2}}{\pi}$ and $x \in [-\pi, \pi]$ is solved using the proposed approach with time step $\frac{1}{300\pi}$. The plot is given in figure 3. The analytic solution is given in [4] with surface plot figure 4. Figure 5 shows the surface plot of proposed approach. Figure 6 gives the root mean square error plot for both classical parabolic and wavelet based proposed approach. The values for solution at different time steps is observed with time step take as $\frac{1}{300\pi}$. Root mean square error plots shows the improvement in solution achieved with less error for proposed method as compared to the classical approach.

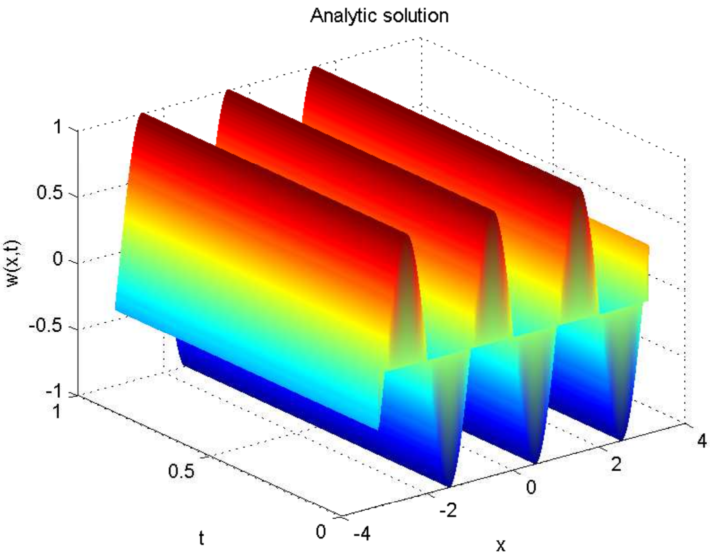

Figure 4: Analytic solution plot for Example 1

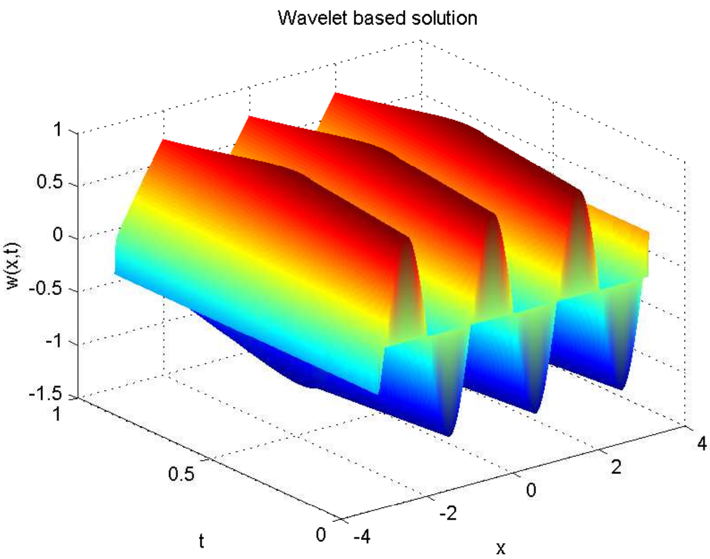

Figure 5: Wavelet based solution plot for Example 1

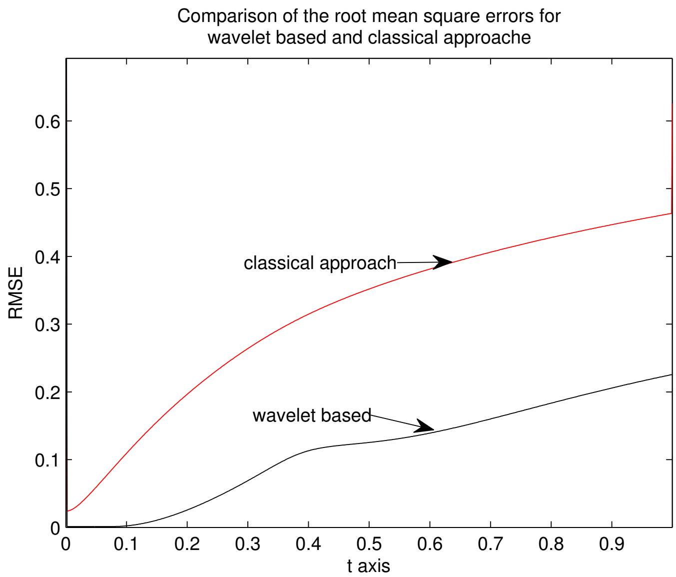

Figure 6: Root mean square error of classical parabolic and wavelet based parabolic solution with analytic value Example 1

5.2 Example 2

Considering the same equation (1) with $\beta = 1$ for $x \in [0, 1.2]$ with initial condition $w(x,1) = \frac{x}{1+\exp(\frac{1}{4\nu}(x^2 - \frac{1}{4}))}$ with boundary conditions $w(0,t) = 0$ and $w(1.2,t) = 0$. The example is solved for $\nu = 0.005$ the step length of x as 0.01 and for time as 0.001. to obtain figure 7.

The analytic solution is given by

$$w(x,t) = \frac{\frac{x}{t}}{1 + (\frac{t}{t_0})^{\frac{1}{2}} \exp(\frac{x^2}{4\nu t})}$$

with $t \geq 1$, with $t_0 = exp(\frac{1}{8\nu})$. The results obtained are compared with results obtained by [20], [21] with courser x grids in table 1.

The figure for Example 2 figure 7 indicates the values at various time using the proposed approach, the values are tabulated and compared with the solutions already known by varying the time and space step sizes. The figure indicates the solution of the proposed method at various time steps to validate the agreement of the obtained and already existing solution in literature.

5.3 Example 3

The equation (1) is solved with $\beta = 1$ with time step 0.001, $w(x,0) = 4x(1-x)$ and boundary conditions $w(0,t) = 0$ and $w(1,t) = 0$ x step length as 0.025 and $\nu = 0.01$.

The analytic solution is given by

Table 1: Comparison of Solution using numerical and analytic solution for $\triangle x = 0.01, \triangle t = 0.001, \nu = 0.005$ for Example 2

x	t	Shu [21] $\triangle t = 0.01$, $\triangle x = 0.0001$	Mittal [20] $\triangle t = 0.001$ $\triangle x = 0.005$	Proposed $\triangle t = 0.001$ $\triangle x = 0.05$	Proposed $\triangle t = 0.001$, $\triangle x = 0.01$	Analytic Solution
0.2	1.7	0.1176565	0.1176452	0.1176093	0.1173452	0.1176452
	2.5	0.0800527	0.079999	0.0799701	0.0797990	0.079999
	3.0	0.0667147	0.066665	0.066641	0.0664658	0.0666658
	3.5	0.0571820	0.05714	0.057122	0.0571422	0.0571422
0.4	1.7	0.2332111	0.235169	0.234525	0.235169	0.2351677
	2.5	0.1591735	0.159977	0.159964	0.1599214	0.1599769
	3.0	0.1328314	0.133321	0.133289	0.1332738	0.1333209
	3.5	0.1139606	0.114278	0.1142459	0.1142380	0.1142779
0.6	1.7	0.2940048	0295857	0.2816436	0.2964038	0.2959097
	2.5	0.2347876	0238129	0.2422683	0.2382074	0.2381207
	3.0	0.1973222	0.199483	0.1997065	0.1994634	0.1994805
	3.5	0.1697753	0.171225	0.1712523	0.1711861	0.1712242
0.8	1.7	0.0008917	0.000638	0.0019488	0.0006381	0.0006465
	2.5	0.1103866	0.102132	0.0931274	0.1016980	0.1020957
	3.0	0.2088346	0.208803	0.2021463	0.2088032	0.2088359
	3.5	0.2119293	0.214593	0.222798	0.2145838	0.2145869

$$w(x,t) = \frac{2\pi\nu \sum_{n=1}^{\infty} a_n \exp^{-n^2\pi^2\nu t} n \sin n\pi x}{a_0 + \sum_{n=1}^{\infty} a_n \exp^{-n^2\pi^2\nu t} n \cos n\pi x} \tag{9}$$

Table 2 gives the comparative results for our approach with results in [7], [3], and they are quiet satisfactory. Figure 8 indicates the proposed solution for example 3 at $\triangle x = 0.025$, $\triangle t = 0.001$, $t \leq 3$ and $\nu = 0.01$.

Convergence analysis for the proposed approach The experimental order of convergence is formulated for the examples computed using the formula

$$EOC^{\triangle x_1 \triangle x_2} = \frac{\log(\varepsilon^{\triangle x_1}) - \log(\varepsilon^{\triangle x_2})}{\log(\triangle x_1) - \log(\triangle x_2)} \tag{10}$$

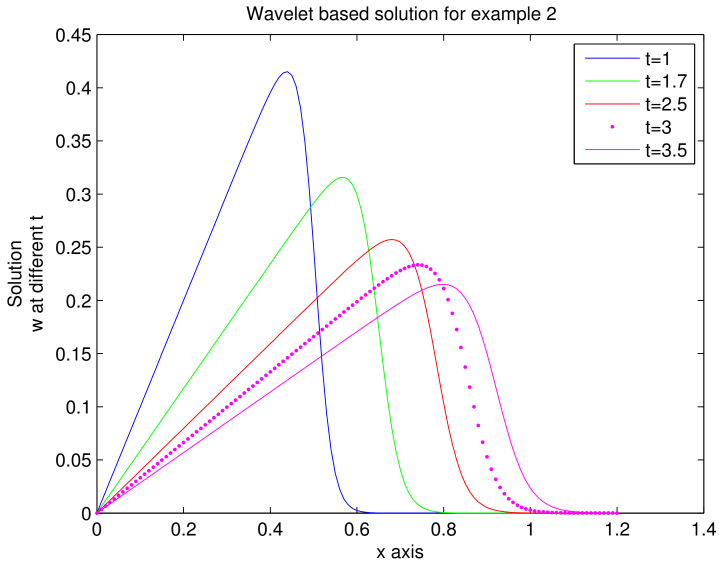

Figure 7: Solution for $\nu = 0.005$, $\triangle t = 0.001$ between 0 and 1 for $t \leq 3.6$ for Example 2

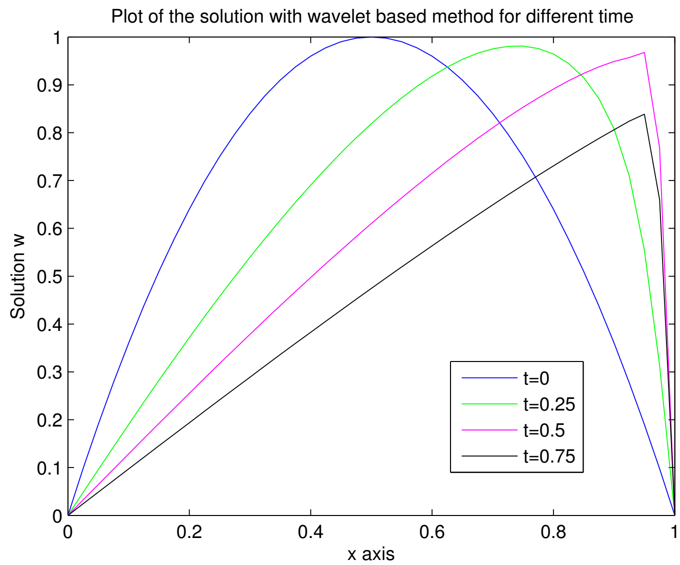

Figure 8: Solution for $\triangle x = 0.025$, $\triangle t = 0.001$, $t \leq 3$ and $\nu = 0.01$. for Example 3

Table 2: Comparison of Solution using numerical and analytic solution for $\triangle x = 0.025, \triangle t = 0.001$ for proposed method, $\nu = 0.01$ for Example 3

x	t	Asai [3] $\triangle x = 0.0125$ $\triangle t = 0.0001$	Aksan[7] $\triangle x = 0.025$ $\triangle t = 0.0001$	Proposed $\triangle x = 0.025$ $\triangle t = 0.001$	Analytic Solution
0.25	0.4	0.36232	0.36225	0.362356	0.3622
	0.6	0.28209	0.28199	0.2820722	0.28204
	0.8	0.23049	0.23039	0.2304514	0.23045
	1.0	0.19472	0.19463	0.1946725	0.19469
	3.0	0.07614	0.07611	0.07611154	0.07613
0.50	0.4	0.68380	0.68371	0.6839751	0.68368
	0.6	0.54840	0.54835	0.548424	0.54832
	0.8	0.45377	0.45374	0.453692	0.45371
	1.0	0.38572	0.38568	0.3856223	0.38568
	3.0	0.15219	0.15216	0.152134	0.15218
0.75	0.4	0.92101	0.92047	0.9219043	0.92050
	0.6	0.78324	0.78302	0.783568	0.78299
	0.8	0.66285	0.66276	0.662885	0.66272
	1.0	0.56940	0.56936	0.5693176	0.56932
	3.0	0.22786	0.22773	0.2277426	0.22774

where $\triangle x_1$ and $\triangle x_2$ are the different mesh sizes. Here

$$\varepsilon^{\triangle x_1} = 100\frac{\|w^{\triangle x_1} - w^{exact}\|_{L_1}}{\|w^{exact}\|_{L_1}} \tag{11}$$

which gives the relative percentage error for the different mesh considered as in table 3, 4. Here w^{exact} indicates the exact solution as a reference value to compute the experimental order of convergence , which is shown to be increasing with increased cell numbers. In table 4 for example 2 the comparison of EOC for both classical finite volume and proposed approach is given which indicates an accelerated value of EOC with the classical approach. The experimental order of accuracy in table 3 for example 1 indicates the rate of convergence increasing with increased grid. The table 4 for example 2 compares the experimental order of convergence in classical finite volume and proposed approach. It indicates an improvement in the rate of convergence with increased grid size for the proposed approach.

Table 3: The experimental order of convergence as per equation 10 for Example 1

No of cells	$\varepsilon^{\triangle x}$	EOC
5	29.9071	
10	29.5381	0.0179
16	13.9055	1.6031
32	12.8645	0.1122

Table 4: The experimental order of convergence according to equation 10 for Example 2 at $\triangle t = 0.001$

No of cells	$\varepsilon^{\triangle x}$ for FVM	$\varepsilon^{\triangle x}$ for proposed method	EOC FVM	EOC Proposed Method
10	17.7418	17.7211		
20	5.1822	5.1550	1.7755	1.7814
40	1.2608	1.2236	2.0392	2.0748
80	0.2932	0.2625	2.1044	2.2207

6 Conclusion

In this paper, we develop a wavelet based finite volume method for solving nonlinear Burgers Equation using Daubechies wavelet as basis functions. In the present method we combined the features of localization due to wavelet approximation and the salient feature of finite volume which utilizes conservation laws within cell interface. The flux conservation defined in average sense over each cell contributes in the improvement of accuracy. This method is tested on three test problems, and the root mean square error figure 6 for example 1 clearly indicates the reduced error. The solution by our approach as shown for example 2 in section 5, given in table 1, by using courser x grid $\triangle x = 0.05$ and $\triangle x = 0.01$ compares very well and gives even closer to the analytical solution as compared to the results obtained by other researchers [21], [20]. It is interesting to note in example 3 given in section 5, that we have considered 10 times larger time step than other researchers as in [3],[7], given in table 2 which helps in obtaining the solution at the desired time faster, with comparable solution.

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