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Approximate Solution of Powell-Eyring Fluid Flow Problem using B-spline Collocation

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Abstract

The pseudoplastic fluid which obeys the Powell-Eyring model flows between two vertical parallel plates as a result of buoyancy force has been analyzed. The arising governing equations of momentum and energy are transformed into nonlinear boundary value problem with Dirichlet boundary conditions. The solution is obtained by cubic, quartic and quintic B-spline collocation method and linearize the original non-linear problem quasilinearization technique is used. B-splines are basis functions for piecewise polynomial having high level of derivative continuity. They possess attractive properties for fluid flow problems. The numerical results are compared with the previous published work in the limiting sense.

Keywords: Pseudoplastic fluid, Powell-Eyring model, Quasilinearization technique, B-spline collocation.

AMS Subject Classification: 65D07

1. Introduction

The natural convection behavior of pseudo plastic fluids is of interest to engineers working on the fluid mechanics of non-Newtonian fluids. Fluids which do not obey Newton's law of viscosity are known as non-Newtonian fluids. As in most of the practical cases the fluid under consideration is not simple Newtonian one; there is an ominous need to develop mathematical models for complex fluids, referred as non-Newtonian fluids. For the flow of a specific class of such fluids between vertically standing flat plates natural heat convection has been discussed by Na [12]. Rajagopal and Na [14] presented a comprehensive thermodynamic analysis of constitutive relations for some classes of non-Newtonian fluids; one of them is known as third grade fluid and many researchers have used this constitutive relation to model the flow of non-Newtonian fluids [6,8,9].

In most of the other physical and practical situations the equations describing the flow are non-linear and difficult to find exact solution. For this purpose many numerical schemes have been developed to approximate the solution in better way, one of such techniques is B-spline Collocation Method, which is very effective and easy to apply [3].

Properties of B-spline schemes: (see [13])

i) B-splines functions are essentially characterized by their polynomial order k (k=degree + 1) and the level of derivative continuity enforced across cells of the portioned computational domain. The polynomial order dictates the asymptotic order of accuracy while the level of continuity affects resolving power.

ii) The availability of efficient algorithms (e.g. de Boor,1977) [5] for evaluating B-splines and their derivatives has been a landmark for calculating with B-splines. These algorithms allow the systematic formulation of B-spline schemes of arbitrary order k on nonuniform grids, with k being a mere input parameter.

iii) B-splines have minimal compact support. More precisely, a B-spline function of order k is nonzero over k consecutive grid intervals. This property leads to sparse banded matrix representation of differential operators.

iv) Boundary conditions of various types are easily incorporated into a B-spline basis. Approximation of boundary value problems does not require special treatment at boundaries, as is found necessary for high-order finite difference methods [4].

v) B-spline representation of functions is shape preserving. In particular, it does not exhibit spurious oscillations, also known as the Runge phenomenon, which are usually experienced with global polynomial approximations.

vi) B-splines of maximal continuity, also termed smoothest B-splines, yield numerical schemes with “spectral-like” resolving power, allowing accurate approximation of problems with a broad range of scales [17]. This property is highly attractive for fluid flows in the transitional and turbulent regimes.

This paper is concerned about the behavior of solution of nonlinear boundary value problem arising as a result of buoyancy force when a pseudo plastic fluid, which obeys the Powell-Eyring model, flows between two vertical parallel plates has been analyzed. The governing differential equation is solved by B-spline collocation method. In section 2, the mathematical model of the problem given by Na [12] is presented. The cubic, quartic and quintic B-spline collocation method for linear two point boundary value problem is explained in section 3, 4 and 5 respectively. The results are displayed in tabular and graphical manner in Section 6 and the discussion of results and conclusion are drawn in Section 7 and 8 respectively.

2. Formulation of the Problem:

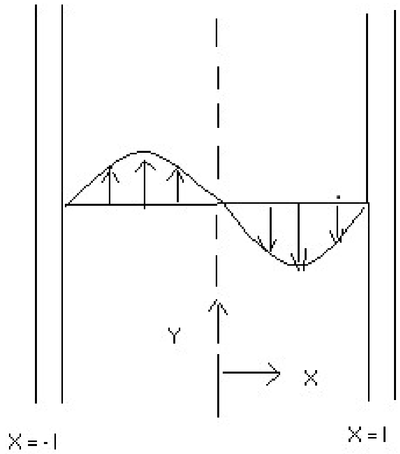

Fig: 1 The Flow System Which Obeys the Powell-Eyring Model

As shown in figure 1, two vertical infinite flat plates parallel to each other. The distance between these two palates is 2 units. Assume that y-axis passes through a line bisecting the distance between these two plates, as shown in figure 1. A non-Newtonian fluid flows between these plates. In the case of vertical plates, pressure in horizontal plane is equal to the gravitational pressure on the fluid and hence pressure on fluid remains constants. The present work is intended to study an incompressible and steady flow of a viscous fluid flowing between two vertical plates. Considering an isothermal case, the boundary force results a flow which is governed by the momentum and energy equations as follows:

$$\frac{d\tau}{dx} + \rho e g (T - T_m) = 0 \tag{1}$$

and

$$\frac{d^2T}{dx^2} = 0 \tag{2}$$

where the well-known parameters appearing in theses equations are presented in usual notations, viz.

ρ	=	density of the fluid
e	=	coefficient of expansion
g	=	gravitational acceleration
T	=	temperature of the fluid
T_m	=	mean temperature of the fluid
τ	=	shearing stress
μ	=	viscosity of the fluid
v	=	velocity of the fluid

Powell – Eyring Fluid:

Let us have a non-Newtonian fluid which obeys Powell-Eyring model. The velocity gradient v and a stress tensor τ are related in this model as given below

$$\tau = \mu \frac{dv}{dx} + \frac{1}{B} \text{arc} \sin h \left(\frac{1}{C}\frac{dv}{dx}\right) \tag{3}$$

where B and C are constants.

Now energy equation (2) gives

$$T = T_m - \tfrac{1}{2}(T_2 - T_m)\, x/1 \tag{4}$$

Let us encounter the following dimensionless variables:

$$\tau^* = \frac{\tau_1^2}{a\mu}, \quad x^* = \frac{x}{1} \quad \text{and} \quad v^* = \frac{v_2}{aN_{pr}N_{gr}} \tag{5}$$

where N_{pr} and N_{gr} are Prandtl number and Grashoff number, respectively. These numbers are defined as

$$N_{gr} = g\rho^2\, el^3\, (T_2 - T_m)/\mu^2$$

$$N_{pr} = \mu/\rho a$$

Substituting dimensionless quantities due to equation (1) in the equation (3) and making proper use of relation (4) for simplifying the equation so obtained an ultimate equation is derived in the form

$$\frac{d^2v^*}{dx^{*2}} - \frac{x^*}{1 + \left\{1/\xi[(\varepsilon \frac{dv^*}{dx^*})^2 + 1]^{1/2}\right\}} = 0 \tag{6}$$

where $\varepsilon = a\, N_{pr}N_{gr}/Cl^2$, $\xi = uBC$, $a = k/\rho C_p$ and C_p is specific heat.

Now from the figure it is observed that the velocity at the points $x = \pm 1$ becomes zero. Also the velocity is seen to be zero at x = 0.

Hence, the boundary conditions are given as

$$v^* = 0 \text{ at } x^* = \pm 1 \tag{7}$$

$$v^* = 0 \text{ at } x^* = 0 \tag{8}$$

By virtue of a transformation $x^* = s - 1$, the relation (6) is reduced to

$$\frac{d^2v^*}{ds^2} = \frac{s-1}{1 + \left\{1/\xi[(\varepsilon \frac{dv^*}{ds})^2 + 1]^{1/2}\right\}} \tag{9}$$

with the corresponding boundary conditions

$$v^* = 0 \text{ at } s = 0 \tag{10}$$

$$v^* = 0 \text{ at } s = 1 \tag{11}$$

Equation (9) represents a nonlinear two point boundary value problem, which is solved using cubic, quartic and quintic B-spline collocation method.

3. Cubic B-spline Collocation Method for Linear Second Order Boundary Value Problem:

Consider linear boundary value problem in the form of

$$a_1(x)u'' + a_2(x)u' + a_3(x)u = f(x) \tag{12}$$

Subject to the boundary conditions

$$u(a) = u_0 \quad , \quad u(b) = u_L \tag{13}$$

Subdivide the interval [a,b] and choose piecewise uniform grid points represented by $\prod : x_0 < x_1 < x_2 < .... < x_n$ such that $x_0 = a, x_n = b$ and $h = \frac{b-a}{n}$. Let $S_3(\prod)$ be the space of cubic spline functions over the partition $\prod$. The cubic B-spline basis functions $\{B_i(x)\}$ for $i = -1, 0, 1, ..., n+1$, including two more points on each side of the partition becomes $\prod : x_{-2} < x_{-1} < ... < x_{n+1} < x_{n+2}$.

Now the cubic B-spline basis function is defined as [7]

$$B_j(x) = \frac{1}{h^3}\begin{cases} (x - x_{j-2})^3 & x \in [x_{j-2}, x_{j-1}) \\ (x - x_{j-2})^3 - 4(x - x_{j-1})^3 & x \in [x_{j-1}, x_j) \\ (x_{j+2} - x)^3 - 4(x_{j+1} - x)^3 & x \in [x_j, x_{j+1}) \\ (x_{j+2} - x)^3 & x \in [x_{j+1}, x_{j+2}) \\ 0 & otherwise \end{cases}$$

To solve the boundary value problem $B_j(x)$, $B_j'(x)$ & $B_j''(x)$ are valuated at nodal points given below.

$$B_j(x_k) = \begin{cases} 4 & if \;\; j = k \\ 1 & if \;\; j - k = \pm 1 \\ 0 & if \;\; j - k = \pm 2 \end{cases} \qquad B_j'(x_k) = \begin{cases} 0 & if \;\; j = k \\ \pm\frac{3}{h} & if \;\; j - k = \pm 1 \\ 0 & if \;\; j - k = \pm 2 \end{cases} \qquad B_j''(x_k) = \begin{cases} -\frac{12}{h^2} & if \;\; j = k \\ \frac{6}{h^2} & if \;\; j - k = \pm 1 \\ 0 & if \;\; j - k = \pm 2 \end{cases}$$

Let

$$u(x) = \sum_{j=-1}^{n+1} C_j B_j(x) \tag{14}$$

be an approximate solution of given boundary value problem (12) where C_j's are unknown coefficients and B_j's are cubic B-spline functions. This approximate solution must satisfy the given boundary value problem at nodal points $x = x_i$, substituting (14) in (12) together with boundary conditions (13)

$$\sum_{j=-1}^{n+1} C_j a_1(x_i) B_j''(x_i) + \sum_{j=-1}^{n+1} C_j a_2(x_i) B_j'(x_i) + \sum_{j=-1}^{n+1} C_j a_3(x_i) B_j(x_i) = f(x_i) \qquad i = 0, 1, ..., n \tag{15}$$

$$\sum_{j=-1}^{n+1} C_j B_j(x_0) = u_0 \tag{16}$$

$$\sum_{j=-1}^{n+1} C_j B_j(x_n) = u_L \tag{17}$$

The value of B-spline functions and their derivatives at nodal points are evaluated and substituting in (15), (16) and (17). Hence we get a system of (n+3) linear equations in (n+3) unknowns and $C_{-1}, C_0, ..., C_9$ are obtained. This system can be written in matrix vector form as

$$AX = B$$

Where $X = [C_{-1}, C_0, ..., C_n, C_{n+1}]^T$

$$B = [u_0, f(x_0), ..., f(x_n), u_L]^T$$

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 \; . \; . \; . \; . & 0 \\ \alpha(x_0) & \beta(x_0) & \gamma(x_0) & 0 \; . \; . \; . \; . & 0 \\ 0 & \alpha(x_1) & \beta(x_1) & \gamma(x_1) \; . \; . \; . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & \alpha(x_n) & \beta(x_n) & \gamma(x_n) \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix}$$

Where $\alpha(x_i) = a_1(x_i)\dfrac{6}{h^2} + a_2(x_i)\dfrac{3}{h} + a_3(x_i)$; $i = 0,1,...,n$

$$\beta(x_i) = a_1(x_i)\left(\frac{-12}{h^2}\right) + a_2(x_i)\left(\frac{0}{h}\right) + 4a_3(x_i) \; ; \; i = 0,1,...,n$$

$$\gamma(x_i) = a_1(x_i)\frac{6}{h^2} + a_2(x_i)\left(\frac{-3}{h}\right) + a_3(x_i) \; ; \; i = 0,1,...,n$$

The co-efficient matrix A is a tridiagonal matrix. Because of this nature of matrix A, the determination of the required quantities becomes simple and consumes less time. The values of these constants ultimately yield the approximate cubic B-spline solution of (12) at nodal points.

4. Quartic B-spline Collocation Method for second order linear boundary value problem

Let $S_4(\prod)$ be the space of quartic spline functions over the uniform partition $\prod : x_0 < x_1 < x_2 < .... < x_n$ such that $x_0 = a, x_n = b$ and $h = \dfrac{b-a}{n}$. The quartic B-spline basis functions $\{B_i(x)\}$ for $i = -1, 0, 1, ..., n+2$, introducing four additional knots on each side of the partition $\prod$ (see [11]). Thus partition becomes $\prod : x_{-4} < x_{-2} < ... < x_{n+3} < x_{n+4}$.

Now the quartic B-spline basis function is defined as [13]

$$B_j(x)=\frac{1}{h^4}\begin{cases}(x-x_{j-3})^4 & x\in[x_{j-3},x_{j-2}]\\ (x-x_{j-3})^4-5(x-x_{j-2})^4 & x\in[x_{j-2},x_{j-1}]\\ (x-x_{j-3})^4-5(x-x_{j-2})^4+10(x-x_{j-1})^4 & x\in[x_{j-1},x_j]\\ (x_{j+2}-x)^4-5(x_{j+1}-x)^4 & x\in[x_j,x_{j+1}]\\ (x_{j+2}-x)^4 & x\in[x_{j+1},x_{j+2}]\\ 0 & otherwise\end{cases}$$

To solve the boundary value problem the value of $B_j(x)$,$B_j'(x)$ & $B_j''(x)$ evaluated at nodal points are given below.

$$B_j(x_k)=\begin{cases}11 \quad if \quad j-k=0, j-k=1\\ 1 \quad if \quad j-k=-1, j-k=2\\ 0 \quad if \quad j-k=-2\end{cases}$$

$$B_j'(x_k)=\begin{cases}\frac{-12}{h},\frac{12}{h} & if \quad j-k=0, j-k=1\\ \frac{-4}{h},\frac{4}{h} & if \quad j-k=-1, j-k=2\\ 0 & if \quad j-k=-2\end{cases}$$

$$B_j''(x_k)=\begin{cases}\frac{-12}{h^2} & if \quad j-k=0, j-k=1\\ \frac{12}{h^2} & if \quad j-k=-1, j-k=2\\ 0 & if \quad j-k=-2\end{cases}$$

Let

$$u(x)=\sum_{j=-1}^{n+2} C_j B_j(x) \tag{18}$$

be the approximate solution of given boundary value problem (12) where C_j's are unknown coefficients and B_j's are quartic B-spline functions. This approximate solution must satisfy the given boundary value problem at nodal points $x=x_i$, substituting (18) in (12) together with boundary conditions (13),

$$\sum_{j=-1}^{n+2} C_j a_1(x_i)B_j''(x_i)+\sum_{j=-1}^{n+2} C_j a_2(x_i)B_j'(x_i)+\sum_{j=-1}^{n+2} C_j a_3(x_i)B_j(x_i)=f(x_i) \qquad i=0,1,...,n \tag{19}$$

$$\sum_{j=-1}^{n+2} C_j B_j(x_0)=u_0 \tag{20}$$

$$\sum_{j=-1}^{n+2} C_j B_j(x_n) = u_L \tag{21}$$

The equation (19) for $i = 0,1,2,...,n$ and boundary conditions (20), (21) lead to the (n+3) linear equations in (n+4) unknowns, so one more equation is needed to solve the system. Hence the midpoint of subinterval is used and it is defined as [12]

$$\Gamma = \{\tau_0 = x_0, \tau_1 = \frac{x_0 + x_1}{2}, ..., \tau_n = \frac{x_{n-1} + x_n}{2}, \tau_{n+1} = x_n\}$$

That Γ includes the midpoint of subintervals and for $x = x_0 + \frac{h}{2}$ the values of $B_j(x)$, $B_j'(x)$ & $B_j''(x)$ defined as follows.

$$B_j(x_k) = \begin{cases} \frac{76}{16} & if \quad j-k=0, j-k=2 \\ \frac{1}{16} & if \quad j-k=-1, j-k=3 \\ \frac{230}{16} & if \quad j-k=1 \\ 0 & if \quad j-k=-2 \end{cases}$$

$$B_j'(x_k) = \begin{cases} -\frac{11}{h}, \frac{11}{h} & if \quad j-k=0, j-k=2 \\ -\frac{1}{2h}, \frac{1}{2h} & if \quad j-k=-1, j-k=3 \\ 0 & if \quad j-k=1, j-k=-2 \end{cases}$$

$$B_j''(x_k) = \begin{cases} \frac{12}{h^2} & if \quad j-k=0, j-k=2 \\ \frac{3}{h^2} & if \quad j-k=-1, j-k=3 \\ -\frac{30}{h^2} & if \quad j-k=1 \\ 0 & if \quad j-k=-2 \end{cases}$$

From (12) at $x = x_0 + \frac{h}{2}$, equation becomes

$$\sum_{j=-1}^{n+2} C_j a_1(x_0 + \frac{h}{2}) B_j''(x_0 + \frac{h}{2}) + \sum_{j=-1}^{n+2} C_j a_2(x_0 + \frac{h}{2}) B_j'(x_0 + \frac{h}{2}) + \sum_{j=-1}^{n+2} C_j a_3(x_0 + \frac{h}{2}) B_j(x_0 + \frac{h}{2}) = f(x_0 + \frac{h}{2})$$

Hence by solving the system of (n+4) linear equations in (n+4) unknowns $C_{-1}, C_0, ..., C_9, C_{10}$ are obtained.

This system can be written in matrix vector form as

$$AX = B$$

Where $X = [C_{-1}, C_0, ..., C_{n+1}, C_{n+2}]^T$

$B = [u_0, f(x_0), f(x_{0+h/2}), ..., f(x_n), u_L,]^T$

$$A = \begin{bmatrix} 1 & 11 & 11 & 1 & 0 & . & . & . & & & & 0 \\ \alpha(x_0) & \beta(x_0) & \gamma(x_0) & \delta(x_0) & 0 & . & . & . & & & & 0 \\ \alpha\left(x_0+\frac{h}{2}\right) & \beta\left(x_0+\frac{h}{2}\right) & \gamma\left(x_0+\frac{h}{2}\right) & \delta\left(x_0+\frac{h}{2}\right) & \omega\left(x_0+\frac{h}{2}\right) & . & . & . & & & & 0 \\ 0 & \alpha(x_1) & \beta(x_1) & \gamma(x_1) & \delta(x_1) & . & . & . & . & . & & 0 \\ . & . & . & . & . & . & & & & & & \\ . & . & . & . & . & . & & & & & & \\ 0 & 0 & . & . & . & . & & & \alpha(x_n) & \beta(x_n) & \gamma(x_n) & \delta(x_n) \\ 0 & 0 & . & . & . & . & & & 1 & 11 & 11 & 1 \end{bmatrix}$$

Where $\alpha(x_i) = a_1(x_i)\left(\frac{12}{h^2}\right) + a_2(x_i)\left(\frac{-4}{h}\right) + a_3(x_i)(1)$; $i = 0,1,...,n$

$$\beta(x_i) = a_1(x_i)\left(\frac{-12}{h^2}\right) + a_2(x_i)\left(\frac{-12}{h}\right) + a_3(x_i)(11); \ i = 0,1,...,n$$

$$\gamma(x_i) = a_1(x_i)\left(\frac{-12}{h^2}\right) + a_2(x_i)\left(\frac{12}{h}\right) + a_3(x_i)(11) \ ; \ i = 0,1,...,n$$

$$\delta(x_i) = a_1(x_i)\left(\frac{12}{h^2}\right) + a_2(x_i)\left(\frac{4}{h}\right) + a_3(x_i)(1) \ ; \ i = 0,1,...,n$$

$$\alpha\left(x_0 + \frac{h}{2}\right) = a_1\left(x_0 + \frac{h}{2}\right)\left(\frac{3}{h^2}\right) + a_2\left(x_0 + \frac{h}{2}\right)\left(\frac{-1}{2h}\right) + a_3\left(x_0 + \frac{h}{2}\right)\left(\frac{1}{16}\right)$$

$$\beta\left(x_0 + \frac{h}{2}\right) = a_1\left(x_0 + \frac{h}{2}\right)\left(\frac{12}{h^2}\right) + a_2\left(x_0 + \frac{h}{2}\right)\left(\frac{-11}{h}\right) + a_3\left(x_0 + \frac{h}{2}\right)\left(\frac{76}{16}\right)$$

$$\gamma\left(x_0 + \frac{h}{2}\right) = a_1\left(x_0 + \frac{h}{2}\right)\left(\frac{-30}{h^2}\right) + a_2\left(x_0 + \frac{h}{2}\right)(0) + a_3\left(x_0 + \frac{h}{2}\right)\left(\frac{230}{16}\right)$$

$$\delta\left(x_0 + \frac{h}{2}\right) = a_1\left(x_0 + \frac{h}{2}\right)\left(\frac{12}{h^2}\right) + a_2\left(x_0 + \frac{h}{2}\right)\left(\frac{11}{h}\right) + a_3\left(x_0 + \frac{h}{2}\right)\left(\frac{76}{16}\right)$$

$$\omega\left(x_0 + \frac{h}{2}\right) = a_1\left(x_0 + \frac{h}{2}\right)\left(\frac{3}{h^2}\right) + a_2\left(x_0 + \frac{h}{2}\right)\left(\frac{1}{2h}\right) + a_3\left(x_0 + \frac{h}{2}\right)\left(\frac{1}{16}\right)$$

The co-efficient matrix is a band matrix of order four. So, it becomes simple, easy to solve and consumes less time . The values of these constants gives approximate solution of (12) at nodal points.

5. Quintic B-spline Collocation Method for second order linear boundary value problem

Let $S_5(\prod)$ be the space of quintic spline functions over the uniform partition $\prod$. The quintic B-spline basis functions $\{B_i(x)\}$ for $i = -2, -1, 0, ..., n+2$,introducing five additional knots on each side of the partition $\prod : x_{-5} < x_{-4} < ... < x_{n+4} < x_{n+5}$ and form a basis for the space $S_5(\prod)$.The quintic B-splines are the unique nonzero splines of smallest compact support with knots at $x_{-5} < x_{-4} < ... < x_{n+4} < x_{n+5}$.Now the quintic B-spline basis function is defined as (see [15]-[16])

$$B_j(x) = \frac{1}{h^5}\begin{cases} (x-x_{j-3})^5 & x \in [x_{j-3}, x_{j-2}] \\ (x-x_{j-3})^5 - 6(x-x_{j-2})^5 & x \in [x_{j-2}, x_{j-1}] \\ (x-x_{j-3})^5 - 6(x-x_{j-2})^5 + 15(x-x_{j-1})^5 & x \in [x_{j-1}, x_j] \\ (x_{j+3}-x)^5 - 6(x_{j+2}-x)^5 + 15(x_{j+1}-x)^5 & x \in [x_j, x_{j+1}] \\ (x_{j+3}-x)^5 - 6(x_{j+2}-x)^5 & x \in [x_{j+1}, x_{j+2}] \\ (x_{j+3}-x)^5 & x \in [x_{j+2}, x_{j+3}] \\ 0 & otherwise \end{cases}$$

To solve the second order boundary value problem the values of $B_j(x)$, $B_j'(x)$, $B_j''(x)$ & $B_i'''(x)$ are evaluated at nodal points given below.

$$B_j(x_k) = \begin{cases} 66 & if \quad j-k=0 \\ 26 & if \quad j-k=\pm 1 \\ 1 & if \quad j-k=\pm 2 \end{cases}$$

$$B_j'(x_k) = \begin{cases} 0 & if \quad j-k=0 \\ \frac{-50}{h}, \frac{50}{h} & if \quad j-k=-1, j-k=1 \\ \frac{-5}{h}, \frac{5}{h} & if \quad j-k=-2, j-k=2 \end{cases}$$

$$B_j''(x_k) = \begin{cases} \frac{-120}{h^2} & if \quad j-k=0 \\ \frac{40}{h^2} & if \quad j-k=\pm 1 \\ \frac{20}{h^2} & if \quad j-k=\pm 2 \end{cases}$$

$$B_j'''(x_k) = \begin{cases} 0 & if \quad j-k=0 \\ \frac{120}{h^3}, \frac{-120}{h^3} & if \quad j-k=-1, j-k=1 \\ \frac{-60}{h^3}, \frac{60}{h^3} & if \quad j-k=-2, j-k=2 \end{cases}$$

Let

$$u(x) = \sum_{j=-2}^{n+2} C_j B_j(x) \tag{22}$$

be the approximate solution of given boundary value problem (12) where C_j's are unknown coefficients and B_j's are quintic B-spline functions. This approximate solution must satisfy the given boundary value problem at nodal points $x = x_i$, substituting (22) in (12) together with boundary conditions (13)

$$\sum_{j=-2}^{n+2} C_j a_1(x_i) B_j''(x_i) + \sum_{j=-2}^{n+2} C_j a_2(x_i) B_j'(x_i) + \sum_{j=-2}^{n+2} C_j a_3(x_i) B_j(x_i) = f(x_i) \qquad i = 0,1,...,n \tag{23}$$

$$\sum_{j=-2}^{n+2} C_j B_j(x_0) = u_0 \tag{24}$$

$$\sum_{j=-2}^{n+2} C_j B_j(x_n) = u_L \tag{25}$$

The equation (23) for $i = 0,1,2,...,n$ and boundary conditions (24), (25) lead to the (n+3) linear equations in (n+5) unknowns. Still two more equations are needed. Differentiating (12), we get

$$a_1(x)u''' + a_2(x)u'' + a_3(x)u' = f'(x)$$

At $x = x_0$

$$\sum_{j=-2}^{n+2} C_j a_1(x_0) B_j'''(x_0) + \sum_{j=-2}^{n+2} C_j a_2(x_0) B_j''(x_0) + \sum_{j=-2}^{n+2} C_j a_3(x_0) B_j'(x_0) = f'(x_0) \tag{26}$$

At $x = x_n$

$$\sum_{j=-2}^{n+2} C_j a_1(x_n) B_j'''(x_n) + \sum_{j=-2}^{n+2} C_j a_2(x_n) B_j''(x_n) + \sum_{j=-2}^{n+2} C_j a_3(x_n) B_j'(x_n) = f'(x_n) \tag{27}$$

Equations (23)-(27) form a system can be written in matrix vector form as

$$AX = B$$

Where $X = [C_{-2}, C_{-1}, C_0, ..., C_{n+1}, C_{n+2}]^T$

$$B = [u_0, f(x_0), f'(x_0), f(x_1), ..., f(x_n), f'(x_n), u_L,]^T$$

$$A = \begin{bmatrix}
1 & 26 & 66 & 26 & 1 & 0 & . & & & & & 0 \\
\alpha(x_0) & \beta(x_0) & \gamma(x_0) & \delta(x_0) & \omega(x_0) & 0 & . & . & . & & & 0 \\
\tilde{\alpha}(x_0) & \tilde{\beta}(x_0) & \tilde{\gamma}(x_0) & \tilde{\delta}(x_0) & \tilde{\omega}(x_0) & 0 & . & . & . & & & 0 \\
0 & \alpha(x_1) & \beta(x_1) & \gamma(x_1) & \delta(x_1) & \omega(x_1) & 0 & . & . & & & 0 \\
. & . & . & . & . & . & & & & & & \\
. & . & . & . & . & . & & & & & & \\
0 & 0 & . & . & . & . & & \alpha(x_n) & \beta(x_n) & \gamma(x_n) & \delta(x_n) & \omega(x_n) \\
0 & 0 & . & . & . & . & & \tilde{\alpha}(x_n) & \tilde{\beta}(x_n) & \tilde{\gamma}(x_n) & \tilde{\delta}(x_n) & \tilde{\omega}(x_n) \\
0 & 0 & & & & & & 1 & 26 & 66 & 26 & 1
\end{bmatrix}$$

Where $\alpha(x_i) = a_1(x_i)\left(\frac{20}{h^2}\right) + a_2(x_i)\left(\frac{-5}{h}\right) + a_3(x_i)(1) \quad ; i = 0,1,...,n$

$$\beta(x_i) = a_1(x_i)\left(\frac{40}{h^2}\right) + a_2(x_i)\left(\frac{-50}{h}\right) + a_3(x_i)(26) \quad ; i = 0,1,...,n$$

$$\gamma(x_i) = a_1(x_i)\left(\frac{-120}{h^2}\right) + a_2(x_i)(0) + a_3(x_i)(66) \quad ; i = 0,1,...,n$$

$$\delta(x_i) = a_1(x_i)\left(\frac{40}{h^2}\right) + a_2(x_i)\left(\frac{50}{h}\right) + a_3(x_i)(26) \quad ; i = 0,1,...,n$$

$$\omega(x_i) = a_1(x_i)\left(\frac{20}{h^2}\right) + a_2(x_i)\left(\frac{5}{h}\right) + a_3(x_i)(1) \quad ; i = 0,1,...,n$$

$$\tilde{\alpha}(x_i) = a_1(x_i)\left(\frac{-60}{h^3}\right) + a_2(x_i)\left(\frac{20}{h^2}\right) + a_3(x_i)\left(\frac{-5}{h}\right) \quad ; i = 0 \ \& \ n$$

$$\tilde{\beta}(x_i) = a_1(x_i)\left(\frac{120}{h^3}\right) + a_2(x_i)\left(\frac{40}{h^2}\right) + a_3(x_i)\left(\frac{-50}{h}\right) \quad ; i = 0 \ \& \ n$$

$$\tilde{\gamma}(x_i) = a_1(x_i)(0) + a_2(x_i)\left(\frac{-120}{h^2}\right) + a_3(x_i)(0) \quad ; i = 0 \ \& \ n$$

$$\tilde{\delta}(x_i) = a_1(x_i)\left(\frac{-120}{h^3}\right) + a_2(x_i)\left(\frac{40}{h^2}\right) + a_3(x_i)\left(\frac{50}{h}\right) \quad ; i = 0 \ \& \ n$$

$$\tilde{\omega}(x_i) = a_1(x_i)\left(\frac{60}{h^3}\right) + a_2(x_i)\left(\frac{20}{h^2}\right) + a_3(x_i)\left(\frac{5}{h}\right) \quad ; i = 0 \ \& \ n$$

Hence by solving the system of (n+5) linear equations in (n+5) unknowns we obtain the approximate quantic B-spline solution of (12) at nodal points.

6. Solution Using B-spline Collocation Method:

To solve nonlinear differential equation (9) together with boundary conditions (10) and (11), first it is to be linearized using quasilinaearization technique [1] and transformed into

$$\frac{d^2v^*(i+1)}{ds^2} - \frac{\varepsilon^2\xi(s-1)\left(\frac{dv^{*(i)}}{ds}\right)s}{\varphi\,(\xi\varphi + 1)^2}\left(\frac{dv^{*(i+1)}}{ds}\right) = \frac{\xi\,(s-1)\,\varphi}{(1+\xi\varphi)} - \frac{\varepsilon^2\xi(s-1)}{\varphi\,(\xi\varphi+1)^2}\left(\frac{dv^{*(i)}}{ds}\right)^2 \tag{28}$$

Where $\varphi = \left[\left(\varepsilon\frac{dv^{*(i)}}{ds}\right)^2 + 1\right]^{1/2}$ and the super script i indicates the number of iteration in the iterative process to be carried out. The boundary conditions are

$$v^{*(i+1)} = 0 \quad \text{at} \quad s = 0 \tag{29}$$

$$v^{*(i+1)} = 0 \quad \text{at} \quad s = 1 \tag{30}$$

In order to obtain a B-spline approximation, let us begin with fitting a straight line $v^{*(0)}(s) = as + b$ substituting and satisfying the conditions (29) and (30), as an initial guess. This straight line is found to be

$$v^{*(0)}(s) = 0 \tag{31}$$

Substituting (21) in (28), we get

$$\frac{d^2 v^{*(i+1)}}{ds^2} = \frac{\xi(S-1)\phi}{(1+\xi\phi)} \tag{32}$$

Taking N=10 equal subintervals of the domain [0, 1] a cubic, quartic and quintic B-spline for this particular problem (32) subject to the boundary conditions (29) and (30) is proceeding as explained in section-3,4 and 5.we obtain the results of equation (9) using quasilinearization and B-spline collocation method. These results are compared with the available results in the following comparison table.

Table I: Comparison of numerical Solution: $\xi = 10,\ \varepsilon = 0.01$

Values of S	Solution by Cubic B-spline collocation	Solution by Quartic B-spline collocation	Solution by Quintic B-spline collocation	Solution by Bickley's Method	Solution by Finite Difference Method
0	0	0	0	0	0
0.2	0.04363	0.04364	0.04365	0.04363	0.0437
0.4	0.05818	0.05818	0.05819	0.05816	0.0582
0.6	0.05091	0.05091	0.05090	0.05088	0.0509
0.8	0.02901	0.02909	0.02907	0.02906	0.02901
1	0	0	0	0	0

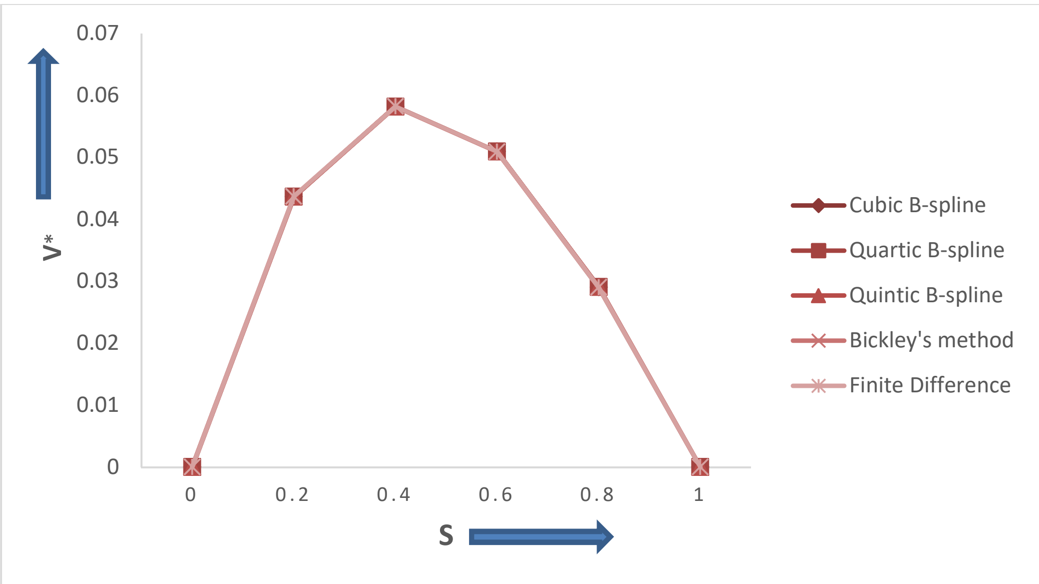

Fig-2: Velocity distributions of Powell-Eyring Fluid: $\xi = 10,\ \varepsilon = 0.01$

7. Discussion of Results:

The Powell Eyring model is governed by a nonlinear two point boundary value problem of second order. The results obtain by cubic, quartic and quintic B-spline collocation method are compared with the solution available in the literature as in Table-1. Fig-2 shows that velocity increases gradually as per the increment in displacement and at a particular point velocity falls down inspite of increasing displacement and becomes zero at the second end point i.e. at $s = 1$. It is also observed that increase of accuracy as the order of spline increases is not much significant in this problem. It means for second order boundary value problem cubic B-spline gives the sufficient result.

8. Conclusion:

The main interest in using B-splines is that they provide grid flexibility, local support and relative to compact schemes the boundary treatment is straightforward and appears to be more robust.

By combination of two processes viz quasilinearization technique and B-spline collocation method, the original non-linear problem can be linearized into a sequence of linear boundary value problem and finally into a linear differential equation which ultimately is a solution of nonlinear equation. The method can easily be applied to similar problems that arise in physical and engineering sciences problems. Also the method is competent enough to solve the likewise nonlinear problems.

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