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Some remarks on the Nash equilibrium for a non-negative matrix

Indrajit Lagu*, Peter Zizler**

Mount Royal University

4825 Mt Royal Gate South West, Calgary, AB T3E 6K6, Canada.

Email: *ilagu@mtroyal.ca, **pzizler@mtroyal.ca

Abstract

Consider a non-negative matrix P of size $n \times n$ which fails to admit a mixed strategy Nash equilibrium. We can consider the mixed Nash equilibria defined on the $n-1 \times n-1$ minors of the matrix P and potentially have a vast number of candidates for the restricted Nash equilibrium. We give an intuitive condition for the choice of minor(s) for P to have an optimal Nash equilibrium restricted solution(s). As an example, we focus on a specific example given in Beyond Numeracy by J. A. Paulos.

Keywords: non-negative matrix, mixed strategy Nash equilibrium, matrix minors, and Lagrange Multipliers

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1 Introduction.

Consider a probabilistic problem where a pitcher throws three types of balls at the batter (fastball, screwball, or curveball). Evidently, the batter does not know what type of throw is coming and therefore he needs to prepare for any of the three. Below is a table that indicates the probabilities of success for the batter depending on the pitcher throw (row) and batter preparation (column). If the pitcher throws a fastball and the batter prepares for screwball there is a 0.3 probability of success for the batter. This is indicated in the $(1, 2)$ entry.

	fastball	screwball	curveball
fastball	0.4	0.3	0.2
screwball	0.2	0.4	0.3
curveball	0.2	0.1	0.4

Naturally, if the batter always gets ready for the screwball the pitcher will always throw curveball and the probability of the batter's success is minimized at 0.1. However, the batter can mix it up. Let d_f, d_s and d_c be the proportions of batter readiness for the types of throws, the subscript indicates the type of throw. We have

$$d_f + d_s + d_c = 1.$$

The pitcher will mix it up as well. Let a_f, a_s and a_c be the proportions of types of balls the pitcher throws, the subscript indicates the type of throw. We have

$$a_f + a_s + a_c = 1.$$

Consider the matrix P (For simplicity, we will illustrate on a general 3×3 matrix as the general case becomes clear.):

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{12} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

Then the expected batter success is given by the quadratic form

$$\langle P\mathbf{d}, \mathbf{a} \rangle \text{ with } \mathbf{d} = (d_f, d_s, d_c)^T \text{ and } \mathbf{a} = (a_f, a_s, a_c)^T,$$

subject to

$$d_f + d_s + d_c = 1 \text{ and } a_f + a_s + a_c = 1.$$

For the general case denote let $\mathbf{d} := (d_1, \ldots, d_n)^T$ and $\mathbf{a} := (a_, \ldots, a_n)^T$. It is not difficult to see

$$\min_{i,j} p_{ij} \leq \langle P\mathbf{d}, \mathbf{a} \rangle \leq \max_{ij} p_{ij}.$$

In our paper we will consider square $n \times n$ matrices P that are *non-negative*, meaning all of its entries are non-negative.

Definition We say that a non-negative matrix P has a mixed strategy Nash equilibrium at the point $(\mathbf{d}_0, \mathbf{a}_0)$ if $F(\mathbf{d}, \mathbf{a}) := \langle P\mathbf{d}, \mathbf{a} \rangle$, subject to $\sum d_i = 1$ and $\sum a_i = 1$, has a stationary point at $(\mathbf{d}_0, \mathbf{a}_0)$ with $\mathbf{d}_0 \geq 0$ and $\mathbf{a}_0 \geq 0$.

One can think of this point as a probability distribution for both the batter and the pitcher where any minor deviations from each does not seem change the outcome for the batter success. There is a vast literature on this subject in the context of mixed strategy Nash equilibrium, we just mention the following references: [2] and [3].

2 Some consequences of known results

Suppose P is nonsingular. Let $Q := P^{-1} = [q_{ij}]$. Let $r_i = \sum_{j=1}^{n} q_{ij}$ and $r_j = \sum_{i=1}^{n} q_{ij}$ be the row and column sums of Q. Finally, let $s := \sum_i r_i = \sum_j r_j$. Let $C = [C_{ij}]$ be the cofactor matrix

$$C_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the minor obtained by removing the ith row and jth column in P. Evidently, $PC^T = \det(P)I$. Let c denote the sum of all signed minors of P (in other words the sum of the entries in C), let c_i be the sum of all entries in the ith row of C^T, and let c_j be the sum of all entries in the jth column of C^T .

Definition. We say that a non-negative matrix P is row (column) full if all of P's row (column) vectors are nonzero.

For convenience we provide the reader a proof for a result that follows as a consequence from the material in the first chapter of [1].

Theorem. Let P be a non-negative square matrix. Then

1. Suppose P is nonsingular then P has a mixed strategy Nash equilibrium if and only if all row and column sums of Q and all row and column sums of C^T are non-negative. (In other words, $r_i, r_j \geq 0$ and $c_i, c_j \geq 0$ for all i, j.)

   In particular, the equilibrium is attained at $\mathbf{d} = (d_1, \ldots, d_n)^T$ and $\mathbf{a} = (a_1, \ldots, a_n)^T$ with $d_i = r_i/s = c_i/c$ and $a_j = r_j/s = c_j/c$. Moreover, if the mixed strategy Nash equilibrium exists then it must be unique with

   $$\lambda := \langle P\mathbf{d}, \mathbf{a} \rangle = \frac{\det(P)}{c}.$$

2. Suppose P is singular and row or column full. Furthermore assume the sum of all minors of P is nonzero. Then P fails to admit a mixed strategy Nash equilibrium.

Proof. Let $F = F(\mathbf{d}, \mathbf{a}) = \langle P\mathbf{d}, \mathbf{a} \rangle$ where $(\mathbf{d}_0, \mathbf{a}_0)$ is a stationary point for F. Using Lagrange multipliers there exist $\lambda, \beta \in \mathbf{R}$

$$\nabla F = \lambda \nabla G_1 + \beta \nabla G_2$$

where

$$G_1 = \sum d_i - 1 \text{ and } G_2 = \sum a_i - 1.$$

These conditions translate to

$$\begin{pmatrix} p_{11} & \cdots & p_{1n} & -1 \\ \vdots & & \vdots & \vdots \\ p_{n1} & \cdots & p_{nn} & -1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_n \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} p_{11} & \cdots & p_{n1} & -1 \\ \vdots & & \vdots & \vdots \\ p_{1n} & \cdots & p_{nn} & -1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Now suppose that P is an invertible matrix. Then there is a $\lambda > 0$ with

$$P\mathbf{d} = \lambda\mathbf{1},$$

for some $\lambda > 0$. Similarly, there is a $\beta > 0$ with

$$P^*\mathbf{a} = \beta\mathbf{1}.$$

Note that P^* is also invertible. Observe that $\lambda = \beta = \langle P\mathbf{d}, \mathbf{a}\rangle$ since

$$\begin{aligned} \lambda &= \lambda \langle \mathbf{1}, \mathbf{a}\rangle \\ &= \langle P\mathbf{d}, \mathbf{a}\rangle \\ &= \langle \mathbf{d}, P^*\mathbf{a}\rangle \\ &= \langle P^*\mathbf{a}, \mathbf{d}\rangle \\ &= \beta \langle \mathbf{1}, \mathbf{d}\rangle \\ &= \beta. \end{aligned}$$

Since $P\mathbf{d} = P^*\mathbf{a}$ we observe

$$\mathbf{a} = (P^*)^{-1} P\mathbf{d} = \lambda Q^*\mathbf{1},$$

thus $a_j = r_j/s_j$. Similarly

$$\mathbf{d} = P^{-1}P^*\mathbf{a} = \lambda Q\mathbf{1},$$

and thus $d_i = r_i/s_i$. Observe that it can not happen that all r_i's are zero. If it were the case then the vector with all ones as entries would be in the kernel of P^{-1} which is impossible. Similarly, not all r_j's can be zero, the vector with all ones as entries would be in the kernel of $(P^*)^{-1}$, which is impossible. Finally observe that

$$P^{-1} = \frac{1}{\det(P)} C^T.$$

Hence part 1 of the theorem.

For part 2, now assume that P in singular. Let

$$A := \begin{pmatrix} p_{11} & \cdots & p_{1n} & -1 \\ \vdots & & \vdots & \vdots \\ p_{n1} & \cdots & p_{nn} & -1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}.$$

Then $\det(A)$ equals to the sum of all minors of P. Suppose now A is nonsingular. Then we must have a unique solution to

$$\begin{pmatrix} p_{11} & \cdots & p_{1n} & -1 \\ \vdots & & \vdots & \vdots \\ p_{n1} & \cdots & p_{nn} & -1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_n \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

with

$$\lambda = \begin{vmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{vmatrix} = 0.$$

Suppose P admits a mixed Nash equilibrium. Then we must have $P\mathbf{d} = 0$. Similar arguments show $P^*\mathbf{a} = 0$ as well. Note $\mathbf{d} \geq 0$ and $\mathbf{a} \geq 0$. Since P is row or column full, we obtain a contradiction.

In the case of P being singular and the matrix P having the sum of all its minors equal to zero as well, the matrix P might admit a mixed strategy Nash equilibrium or not. For example, if we let

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

then P admits a mixed strategy Nash equilibrium at $\mathbf{d} = \mathbf{a} = (1/2, 1/2)^T$ with $\lambda = 1$. On the other hand, if we let

$$P = \begin{pmatrix} 1 & 1 \\ 1/2 & 1/2 \end{pmatrix},$$

then P fails to admit a mixed strategy Nash equilibrium.

3 Some intuitive reasoning

It is not difficult to see that the mixed strategy Nash equilibrium has to be a saddle. One way to see this is to write the explicit expression for the function F and consider its Hessian matrix which then must have zero trace.

We have used Lagrange multipliers in the proof of the result above. This concept has an intuitive explanation. Recall the mixed strategy Nash equilibrium occurs at the vector $\mathbf{d}$ with the property

$$P\mathbf{d} = \lambda\mathbf{1}$$

for some $\lambda \geq 0$. Therefore, at the mixed strategy Nash equilibrium the batter chooses the defense vector $\mathbf{d}$ in such a way that the pitcher gains no benefit form switching among the pure states of fastball, curveball and the screwball. Similarly, we have

$$P^*\mathbf{a} = \lambda\mathbf{1}.$$

This means that at the mixed strategy Nash equilibrium now the pitcher chooses the attack vector $\mathbf{a}$ in such a way that the batter gains no benefit form switching among the pure states of fastball, curveball and the screwball.

4 Generic example

Let

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.4 \end{pmatrix}.$$

The matrix P admits a mixed strategy Nash equilibrium. We have

$$P^{-1} = \begin{pmatrix} 3.8235 & -2.9412 & 0.2941 \\ -0.5882 & 3.5294 & -2.3529 \\ -1.7647 & 0.5882 & 2.9412 \end{pmatrix},$$

yielding

$$\mathbf{d} = \begin{pmatrix} 0.3333 \\ 0.1667 \\ 0.5000 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} 0.4167 \\ 0.3333 \\ 0.2500 \end{pmatrix}$$

and $\lambda = 0.2833$.

5 Condition on minors and the Paulos Example

Paulos gives the following payoff matrix in [4]:

	curveball	fastball	screwball
curveball	0.4	0.3	0.0
fastball	0.2	0.4	0.3
screwball	0.0	0.2	0.4

Paulos' example gives rise to

$$P := \begin{pmatrix} 0.4 & 0.3 & 0.0 \\ 0.2 & 0.4 & 0.3 \\ 0.0 & 0.2 & 0.4 \end{pmatrix}.$$

In this case,

$$P^{-1} = \begin{pmatrix} 6.250 & -7.500 & 5.625 \\ -5.000 & 10.000 & -7.500 \\ 2.500 & -5.000 & 6.250 \end{pmatrix},$$

which fails to have a mixed strategy Nash equilibrium. However, it is possible to restrict ourselves to a 2×2 subcase as Paulos did.

He claims the following 2×2 solution: The pitcher throws curveballs ($a_c = 0.4$) and screwballs ($a_s = 0.6$) and the batter prepares for fastballs ($d_f = 0.8$) and screwballs ($d_s = 0.2$). This corresponds to the following submatrix

$$\text{CSFS} = \begin{pmatrix} 0.3 & 0.0 \\ 0.2 & 0.4 \end{pmatrix}$$

with $\lambda = 0.240$.

However, there are 9 other 2×2 minors. However, there are only 5 with Nash equilibria. These are the following.

- The pitcher throws curveballs ($a_c = 2/3$) and fastballs ($a_f = 1/3$) and the batter prepares for curveballs ($d_c = 1/3$) and fastballs ($d_f = 2/3$). This corresponds to the following submatrix:

$$\text{CFCF} = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.4 \end{pmatrix},$$

  and $\lambda = 1/3$.

- The pitcher throws curveballs ($a_c = 0.2$) and fastballs ($a_f = 0.8$) and the batter prepares for curveballs ($d_c = 0.6$) and screwballs ($d_s = 0.4$). This corresponds to the following submatrix:

$$\text{CFCS} = \begin{pmatrix} 0.4 & 0.0 \\ 0.2 & 0.3 \end{pmatrix},$$

  and $\lambda = 0.240$.

- The pitcher throws curveballs ($a_c = 1/2$) and screwballs ($a_s = 1/2$) and the batter prepares for curveballs ($d_c = 1/2$) and screwballs ($d_s = 1/2$). This corresponds to the following submatrix:

$$\text{CSCS} = \begin{pmatrix} 0.4 & 0.0 \\ 0.0 & 0.4 \end{pmatrix},$$

  and $\lambda = 0.2$.

- The pitcher throws fastballs ($a_f = 2/3$) and screwballs ($a_s = 1/3$) and the batter prepares for fastballs ($d_f = 1/3$) and screwballs ($d_s = 2/3$). This corresponds to the following submatrix:

$$\text{FSFS} = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.4 \end{pmatrix}$$

and we find $\lambda = 1/3$.

There is a way to make the solution in [4] unique. We introduce a concept of a *strong restricted solution*. Let us illustrate on an example. Suppose the pitcher makes an agreement with the batter prior to the game (CSFS case). Imagine that pitcher promises not to throw any fastballs and the batter promises not to prepare for any curveballs, and then either the pitcher or the batter violates the agreement, but not both simultaneously. If the pitcher throws a fastball the payoff for the pitcher (assuming the batter is faithful) is

$$(0.4)(0.8) + (0.3)(0.2) = 0.38 > 0.240,$$

which is not beneficial for the pitcher. Now suppose the batter lies. The payoff for the batter (assuming the pitcher is faithful) is

$$(0.4)(0.4) + (0.0)(0.6) = 0.16 < 0.240,$$

which is not not beneficial for the pitcher either.

Thus if cheating by neither the pitcher nor the batter yields a benefit, while the other remains faithful, then we have a strong restricted 2×2 solution. The original 2×2 solution in [4] is a strong restricted solution. We now show the remaining possible 2×2 solutions above are not strong restricted.

1. The solution CFCF is not strong: if the pitcher throws a screwball then the payoff is

$$\frac{1}{3}(0) + \frac{2}{3}(0.2) = \frac{0.4}{3} < \frac{1}{3},$$

   which benefits the pitcher (assuming a faithful batter). So the pitcher can gain by cheating.

2. The solution CFCS is not strong: if pitcher throws a screwball then the payoff is

$$(0.6)(0) + (0.4)(0.4) = 0.16 < 0.240,$$

   which benefits the pitcher (assuming a faithful batter). So the pitcher can gain by cheating.

3. The solution CSCS is not strong: if the batter prepares for fastball then the payoff is

$$(0.3)(0.5) + (0.2)(0.5) = 0.25 > 0.2,$$

which benefits the batter (assuming a faithful pitcher). So the batter can gain by cheating.

4. The solution FSFS is not strong: if the pitcher throws a curveball the payoff is

$$(0.3)\frac{1}{3} + (0)\frac{2}{3} = 0.1 < \frac{1}{3}$$

which benefits the pitcher (assuming a faithful batter). So the pitcher can gain by cheating.

The generalization to the $n \times n$ P matrix case with the $n-1 \times n-1$ minors can be readily done.

References

[1] Bapat and Rhagavan (1997) *non-negative Matrix Theory and its Applications* Cambridge University Press

[2] Nash, John (1951), "Non-Cooperative Games" The Annals of Mathematics 54(2):286-295.

[3] Osborne, Martin (2004), An introduction to game theory, Oxford University, ISBN 978-0-19-512895-6. Introduction to Nash equilibrium.

[4] J.A. Paulos (1992), Beyond Numeracy, Vintage Books