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ON THE MULTIPLICATIVITY FACTOR AND QUADRATIVITY FACTOR OF A NONNEGATIVE FUNCTION ON A SEMIGROUP

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Abstract. Let ω be a nonnegative function on a semigroup S. Then ω has a multiplicativity factor if there is $\mu > 0$ such that $\omega(st) \leq \mu\omega(s)\omega(t)$ for all $s, t \in S$; ω has quadrativity factor if there is $\lambda > 0$ such that $\omega(s^2) \leq \lambda\omega(s)^2$ for all $s \in S$. Given a nonnegative function ω on a semigroup S, we shall derive necessary and sufficient condition for ω to have a multiplicativity factor. We shall also do the same for quadrativity factor.

1. Introduction

The paper inspired by the papers [1, 2, 3, 4, 5] of R. Arens, M. Goldburg and W. A. J. Luxemburg. Let T be a normed space seminorm on a complex algebra $\mathcal{A}$, i.e., $T(x) \geq 0$, $T(x + y) \leq T(x) + T(y)$ and $T(\alpha x) = |\alpha|T(x)$ for all $x, y \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. A seminorm T on an algebra $\mathcal{A}$ has a *multiplicativity factor (M- factor)* [3, 5] if there is $\mu > 0$ such that $T(xy) \leq \mu T(x)T(y)$ for all $x, y \in \mathcal{A}$; T has a *quadrativity factor (Q- factor)* if there is $\lambda > 0$ such that $T(x^2) \leq \lambda T(x)^2$ for all $x \in \mathcal{A}$. In [5] they derived necessary and sufficient condition for a seminorm to have a multiplicativity factor. In a subsequent paper [3] they derived necessary and sufficient condition for a seminorm to have a quadrativity factor. Let ω be a nonnegative function on a semigroup S. We shall derive necessary and sufficient for a nonnegative function on a semigroup to have a multiplicativity factor as well as a quadrativity factor.

2. Multiplicative factors

Definition 2.1. *Let S be a semigroup, and let ω be a nonnegative function on S. Then*

(i) *ω has a* multiplicativity factor (M- factor) *if there is $\mu > 0$ such that $\omega(st) \leq \mu\omega(s)\omega(t)$ for all $s, t, \in S$.*

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(ii) *ω has a* quadrativity factor (Q- factor) *if there is $\lambda > 0$ such that $\omega(s^2) \leq \lambda\omega(s)^2$ for all $s \in S$.*

(iii) *ω is a* semiweight *if $\omega(st) \leq \omega(s)\omega(t)$ for all $s \in S$.*

(iv) *ω is a* weight *if ω is a semiweight and $\omega(s) > 0$ for all $s \in S$.*

It follows from above definition that if ω has a multiplicativity factor, then it has a quadrativity factor. Also, if ω is a weight, then ω is a semiweight. A semiweight ω on a semigroup S is *proper* if it not identically zero and $\omega(s) = 0$ for some $s \in S$.

Definition 2.2. *Let S be a semigroup. A subset I is a* semigroup ideal *if either $I = \emptyset$ or $(IS \cup SI) \subset I$.*

Definition 2.3. *Let S be a semigroup, and let $\omega : S \to [0, \infty)$ be a map. Then the set $\{s \in S : \omega(s) = 0\}$ is the* kernel of ω *and it is denoted by* $\ker\omega$.

The following characterizes seminorms having multiplicativity factor [5].

Theorem 2.4. [5, Theorem 2.4] *Let $\mathcal{A}$ be an algebra, and let $T \neq 0$ be a seminorm on $\mathcal{A}$. Then*

(i) *T has a multiplicativity factor if and only if $\mathcal{K} = \ker T$ is an ideal in $\mathcal{A}$ and*

$$\mu_{\inf} = \sup\{T(xy) : x, y \in \mathcal{A}, T(x) = 1 = T(y)\} < \infty.$$

(ii) *If T has a multiplicativity factor and $\mu_{\inf} > 0$, then μ is a multiplicativity factor if and only if $\mu \geq \mu_{\inf}$.*

(iii) *If T has a multiplicativity factor and $\mu_{\inf} = 0$, then μ is a multiplicativity factor if and only if $\mu > 0$.*

We have the following analogous result.

Theorem 2.5. *Let S be a semigroup, and let $\omega : S \to [0, \infty)$ be a map. Then the following statements hold.*

(i) *ω has a multiplicativity factor if and only if $\ker\omega$ is an ideal in S and*

$$\mu_{\inf} = \sup\left\{\frac{\omega(st)}{\omega(s)\omega(t)} : x, y \in S, \omega(x) \neq 0 \neq \omega(y)\right\} < \infty.$$

(ii) *Suppose that ω has a multiplicativity factor. Then a constant $\mu > 0$ is a multiplicativity factor for ω if and only if $\mu \geq \mu_{\inf}$.*

Proof. (i) Assume that ω has a multiplicativity factor. Then there is $k > 0$ such that $\omega(st) \leq k\omega(s)\omega(t)$ for all $s, t \in S$. So, if $s, t \in S$, $\omega(s) \neq 0$ and $\omega(t) \neq 0$, then $\frac{\omega(st)}{\omega(s)\omega(t)} \leq k$. Therefore

$$\sup\left\{\frac{\omega(st)}{\omega(s)\omega(t)} : s, t \in S, \omega(s) \neq 0, \omega(t) \neq 0\right\} < \infty.$$

If $s \in S$ and $t \in \ker\omega$, then $\omega(st) \leq k\omega(s)\omega(t)$ implies that $\omega(st) = 0$, i.e., $st \in \ker\omega$. Also, $ts \in \ker\omega$. Therefore $\ker\omega$ is an ideal in S.

Conversely, assume that

$$\mu_{\text{inf}} = \sup\left\{\frac{\omega(st)}{\omega(s)\omega(t)} : s, t \in S, \omega(s) \neq 0, \omega(t) \neq 0\right\} < \infty$$

and $\ker\omega$ is an ideal in S. Let $s, t \in S$. If any of $\omega(s)$ or $\omega(t)$ is zero, then $\omega(st) = 0$ as $\ker\omega$ is an ideal in S. This gives $\omega(st) \leq \mu_{\text{inf}}\omega(s)\omega(t)$. Let $\omega(s) \neq 0$ and $\omega(t) \neq 0$. Then $\frac{\omega(st)}{\omega(s)\omega(t)} \leq \mu_{\text{inf}}$. Therefore $\omega(st) \leq \mu_{\text{inf}}\omega(s)\omega(t)$ for all $s, t \in S$. So, ω has a multiplicativity factor.

(ii) Let μ be a multiplicativity factor for ω. Then $\frac{\omega(st)}{\omega(s)\omega(t)} \leq \mu$ whenever $s, t \in S$, $\omega(s) \neq 0$ and $\omega(t) \neq 0$. But then $\mu \geq \mu_{\text{inf}}$.

Conversely, assume that $\mu \geq \mu_{\text{inf}}$. Let $s, t \in S$. If any of $\omega(s)$ and $\omega(t)$ is zero, then $\omega(st) = 0$ as $\ker\omega$ is an ideal in S. So, $\omega(st) \leq \mu\omega(s)\omega(t)$ in this case. Let $\omega(s) \neq 0$ and $\omega(t) \neq 0$. Then $\omega(st) \leq \mu_{\text{inf}}\omega(s)\omega(t) \leq \mu\omega(s)\omega(t)$. Thus μ is a multiplicativity for ω. □

Corollary 2.6. *Let S be a semigroup, and let ω be a nonzero function on S such that $\omega(s) \geq 0$ for all $s \in S$. Then ω has a multiplicativity factor and $\mu_{\text{inf}} = \sup\{\frac{\omega(st)}{\omega(s)\omega(t)} : s, t \in S, \omega(s) \neq 0, \omega(t) \neq 0\} = 0$ if and only if $st \in \ker\omega$ for all $s, t \in S$.*

Corollary 2.7. *Let S be a semigroup, and let ω be a positive function on S. Then the following statements hold.*

(i) *ω has a multiplicativity factor if and only if $\mu_{\text{inf}} = \sup\{\frac{\omega(st)}{\omega(s)\omega(t)} : s, t \in S\} < \infty$.*

(ii) *Suppose that ω has a multiplicativity factor. Then a real number μ is a multiplicativity factor for ω if and only if $\mu \geq \mu_{\text{inf}}$.*

Corollary 2.8. *If S is a finite semigroup, and if ω is positive function on S, then ω has a multiplicativity factor.*

Proof. Since S is a finite set, $\mu_{\text{inf}} = \sup\{\frac{\omega(st)}{\omega(s)\omega(t)} : s, t \in S\} < \infty$. Therefore ω has a multiplicativity factor. □

Lemma 2.9. *Let S be a semigroup, and let ω be a nonnegative function on S. Suppose that ω has a multiplicativity factor. Then $\ker\omega$ is an ideal in S. If S is a topological semigroup and if ω is continuous, then $\ker\omega$ is a closed ideal in S.*

Recall that a semigroup is *simple* if it has no nontrivial proper ideals.

Corollary 2.10.

(i) *If S is a simple semigroup, then there are no multiplicative proper semiweights on S.*

(ii) *If S is a topological semigroup that has no proper closed ideals, then there are no continuous proper semiweights on S.*

We recall the *Rees quotient* of a semigroup S by a semigroup ideal I. The relation $\sim$ in S, defined by $s \sim t$ if either $s = t$ or both s and t are in I, is an equivalence relation in S. The equivalence classes under $\sim$ are the singleton sets $\{s\}$ with $s \in S \setminus I$ and the set I. Since I is an ideal is S, the relation $\sim$ is a congruence on S. The quotient semigroup S/I is the Rees factor semigroup of S modulo I [6].

Proposition 2.11. *Let ω be a nonnegative function on a semigroup S such that $\ker\omega$ is an ideal in S. Define $\widetilde{\omega} : S/\ker\omega \to [0,\infty)$ by $\widetilde{\omega}([s]) = \omega(s)$ if $s \in S \setminus \ker\omega$ and $\widetilde{\omega}(\ker\omega) = 0$. Then ω has a multiplicative factor if and only if $\widetilde{\omega}$ has a multiplicative factor. In this case, $\mu_{\inf} = \widetilde{\mu}_{\inf}$.*

Proof. Let $\mu > 0$ be a multiplicative factor for ω. Let $[s],[t] \in S/\ker\omega$. If any of $[s]$ and $[t]$ is $\ker\omega$, then $[st] = \ker\omega$ and hence both $\widetilde{\omega}([s][t])$ and $\widetilde{\omega}([s])\widetilde{\omega}([t])$ are zero. Now assume that none of $[s]$ and $[t]$ is $\ker\omega$. Then $\widetilde{\omega}([s][t]) = \widetilde{\omega}([st]) = \omega(st) \le \mu\omega(s)\omega(t) = \mu\widetilde{\omega}([s])\widetilde{\omega}([t])$. Therefore μ is a multiplicativity factor for $\widetilde{\omega}$.

Let μ be a multiplicativity factor for $\widetilde{\omega}$. Let $s,t \in S$. If any of s and t is in $\ker\omega$, then $\omega(st) = 0$ and $\omega(s)\omega(t) = 0$. So, $\omega(st) \le \mu\omega(s)\omega(t)$. Now assume that none of s and t is in $\ker\omega$. If $st \in \ker\omega$, then clearly $\omega(st) \le \mu\omega(s)\omega(t)$ holds. Let $st \notin \ker\omega$. Then $\omega(st) = \widetilde{\omega}([st]) \le \mu\widetilde{\omega}([s])\widetilde{\omega}([t]) = \mu\omega(s)\omega(t)$. Therefore ω has a multiplicativity factor.

It follows from above that μ is a multiplicativity factor for ω if and only if μ is a multiplicativity factor for $\widetilde{\omega}$. Thus $\mu_{\inf} = \widetilde{\mu}_{\inf}$. □

The following is analogous to [5, Theorem 1.2].

Theorem 2.12. *Let ω_1 and ω_2 be nonnegative functions on S, and let ω_2 be submultiplicative, i.e., $\omega_2(st) \le \omega_2(s)\omega_2(t)$ for all $s,t \in S$. Let $\tau \ge \sigma > 0$ be constants such that*

$$\sigma\omega_2(s) \le \omega_1(s) \le \tau\omega_2(s) \qquad (s \in S).$$

If $\mu \geq \frac{\tau}{\sigma^2}$, then μ is a multiplicativity factor for ω_1.

Proof. Let μ satisfy $\mu \geq \frac{\tau}{\sigma^2}$. Let $s, t \in S$. Then

$$\begin{aligned} \mu\omega_1(st) &\leq \mu\tau\omega_2(st) \\ &\leq \mu\tau\omega_2(s)\omega_2(t) \\ &\leq \mu\frac{\tau}{\sigma^2}\omega_1(s)\omega_1(t). \end{aligned}$$

Therefore ω_1 has a multiplicativity factor. □

Corollary 2.13. *Let ω_1 and ω_2 be nonnegative functions on a semigroup S, and let ω_1 and ω_2 be equivalent, i.e., there are positive constants σ and τ such that $\tau\omega_1 \leq \omega_2 \leq \sigma\omega_1$. Then ω_1 has a multiplicative factor if and only if ω_2 has a multiplicative factor.*

Let T be a seminorm on an algerba $\mathcal{A}$, and let T have a multiplicativity factor. Given $c \in \mathcal{A}$, define $T_c(x) = T(cx)$ for all $x \in \mathcal{A}$. It is proved in [5, Theorem 3.1] that T_c has a multiplicative factor if certain conditions are satisfied. We have a similar result in our case.

Theorem 2.14. *Let S be a semigroup, $c \in S$, and let ω have multiplicative factors. Define $\omega_c(s) = \omega(cs)$ for all $s \in S$. Then ω_c has a multiplicative factor if any of the following conditions is satisfied.*

(i) *S has unit and c is invertible.*

(ii) *c is in the center of S and $c = c^2 d$ for some $d \in S$.*

Proof. (i) Let $s, t \in S$. Then

$$\begin{aligned} \omega_c(st) &= \omega(cst) \\ &= \omega(csc^{-1}ct) \\ &\leq \mu\omega(cs)\omega(c^{-1}ct) \\ &\leq \mu^2\omega_c(s)\omega(c^{-1})\omega_c(t) \\ &= \mu^2\omega(c^{-1})\omega_c(s)\omega_c(t) \end{aligned}$$

Therefore ω_c has a multiplicative factor.

(ii) Let $s, t \in S$. Then

$$\begin{aligned} \omega_c(st) &= \omega(cst) \\ &= \omega(c^2 dst) \end{aligned}$$

$$
\begin{aligned}
&= \omega(dcsct) \\
&\leq \mu^2\omega(d)\omega(cs)\omega(ct) \\
&= \mu^2\omega(d)\omega_c(s)\omega_c(t).
\end{aligned}
$$

Therefore ω_c has a multiplicativity factor. □

3. Quadrarive factors

We recall that a nonnegative map ω on a semigroup S has a quadrative factor if there is $\lambda > 0$ such that $\omega(s^2) \leq \lambda\omega(s)^2$ for all $s \in S$. The following theorem characterizes seminorms having quadrativity factor [3].

Theorem 3.1. [3, Theorem 1.2] *Let T be a seminorm on an algebra $\mathcal{A}$. Then*

(i) *T has a quadrativity factor if and only if $\mathcal{K} = \ker T$ is closed under squaring (i.e., $x^2 \in \mathcal{K}$ if $x \in \mathcal{K}$) and*

$$\lambda_{\text{inf}} = \sup\{T(x^2) : x \in \mathcal{A}, T(x) \leq 1\} < \infty.$$

(ii) *If T has a quadrativity factor and $\lambda_{\text{inf}} > 0$, then λ_{inf} is the best (least) quadrativity factor for T.*

(iii) *If T has a quadrativity factor and $\lambda_{\text{inf}} = 0$, then λ is a quadrativity factor if and only if $\lambda > 0$.*

The following is a similar result.

Theorem 3.2. *Let S be a semigroup, and let ω be a nonnegative function on S. Then the following statements hold.*

(i) *ω has a Q- factor if and only if $\ker\omega$ is closed under squaring, i.e., $s^2 \in \ker\omega$ whenever $s \in S$, and $\lambda_{\text{inf}} = \sup\{\frac{\omega(s^2)}{\omega(s)^2} : s \in S, \omega(s) \neq 0\} < \infty$.*

(ii) *Assume that ω has a Q- factor and $\lambda_{\text{inf}} > 0$. Then $\lambda > 0$ is a Q- factor for ω if and only if $\lambda \geq \lambda_{\text{inf}}$.*

(iii) *Assume that ω has a Q- factor and $\lambda_{\text{inf}} = 0$. Then λ is a Q- factor if and only if $\lambda > 0$.*

Proof. (i) Assume that ω has a Q- factor. Then there is $\lambda > 0$ such that $\omega(s^2) \leq \lambda\omega(s)^2$ for all $s \in S$. Let $s \in \ker\omega$. Then $\omega(s^2) \leq \lambda\omega(s)^2 = 0$, i.e., $s^2 \in \ker\omega$. Therefore $\ker\omega$ is closed under squaring. If $s \in S \setminus \ker\omega$, then $\frac{\omega(s^2)}{\omega(s)^2} \leq \lambda$. Therefore $\lambda_{\text{inf}} < \infty$.

Conversely, assume that $\ker\omega$ is closed under squaring and $\lambda_{\text{inf}} < \infty$. Take any $\lambda > \lambda_{\text{inf}}$. If $s \in S \setminus \ker\omega$, then $\omega(s^2) \leq \lambda_{\text{inf}}\omega(s)^2 \leq \lambda\omega(s)^2$. If $s \in \ker\omega$, then clearly $\omega(s^2) \leq \lambda\omega(s)^2$ as $\ker\omega$

is closed under squaring. Thus ω has a Q- factor.

(ii) Let $\lambda \geq \lambda_{\text{inf}}$. By (i), $\ker\omega$ is closed under squaring. So, if $s \in \ker\omega$, then $\omega(s^2) \leq \lambda\omega(s)^2$. Let $s \in S \setminus \ker\omega$. Then $\omega(s^2) \leq \lambda_{\text{inf}}\omega(s)^2 \leq \lambda\omega(s)^2$. Thus λ is a Q- factor for ω.

Conversely, assume that λ is a Q- factor for ω. Then $\frac{\omega(s^2)}{\omega(s)^2} \leq \lambda$ for all $s \in S \setminus \ker\omega$. Therefore $\lambda_{\text{inf}} \leq \lambda$.

(iii) follows from (ii) □

Lemma 3.3. *Let S be a semigroup, let c be in the centre of S and $c = c^2$. If a nonnegative function ω on S has a Q- factor, then the map $\omega_c(s) = \omega(cs)$ $(s \in S)$ has a Q- factor.*

Proof. Let $\lambda > 0$ be a Q- factor for ω. Let $s \in S$. Then

$$\omega_c(s^2) = \omega(cs^2) = \omega(c^2s^2) = \omega((cs)^2) \leq \lambda\omega(cs)^2 = \lambda\omega_c(s)^2.$$

Therefore ω_c has a Q- factor. □

Lemma 3.4. *Let ω_1 and ω_2 be nonnegative functions on S, and let ω_2 satisfy $\omega_2(s^2) \leq \omega_2(s)^2$ for all $s \in S$. Let $\tau \geq \sigma > 0$ be constants such that*

$$\sigma\omega_2(s) \leq \omega_1(s) \leq \tau\omega_2(s) \qquad (s \in S).$$

If $\lambda \geq \frac{\tau}{\sigma^2}$, then λ is a quadrativity factor for ω_1.

Proof. Let λ satisfy $\lambda \geq \frac{\tau}{\sigma^2}$. Let $s \in S$. Then

$$\begin{aligned}\lambda\omega_1(s^2) &\leq \lambda\tau\omega_2(s^2)\\ &\leq \lambda\tau\omega_2(s)^2\\ &\leq \lambda\frac{\tau}{\sigma^2}\omega_1(s)^2.\end{aligned}$$

This gives $\omega_1(s^2) \leq \frac{\tau}{\sigma^2}\omega_1(s)^2$. Therefore ω_1 has a Q factor. □

Corollary 3.5. *Let ω_1 and ω_2 be nonnegative functions on a semigroup S, and let ω_1 and ω_2 be equivalent. Then ω_1 has a Q- factor if and only if ω_2 has a Q- factor.*

Proof. Since ω_1 and ω_2 are equivalent, there are positive constants σ and τ such that $\tau\omega_1(s) \leq \omega_2(s) \leq \sigma\omega_1(s)$ for all $s \in S$. Assume that ω_1 has a Q- factor, say, λ. Let $s \in S$. Then $\omega_2(s^2) \leq \sigma\omega_1(s^2) \leq \sigma\lambda\omega_1(s)^2 \leq \frac{\sigma\lambda}{\tau^2}\omega_2(s)^2$. Therefore ω_2 has a Q- factor. If ω_2 has a Q- factor, then it follows from above arguments that ω_1 has a Q- factor. □

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