ON THE MULTIPLICATIVITY FACTOR AND QUADRATIVITY FACTOR OF A NONNEGATIVE FUNCTION ON A SEMIGROUP

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Abstract. Let $\omega$ be a nonnegative function on a semigroup $S$. Then $\omega$ has a multiplicativity factor if there is $\mu > 0$ such that $\omega(st) \leq \mu \omega(s)\omega(t)$ for all $s,t \in S$; $\omega$ has quadrativity factor if there is $\lambda > 0$ such that $\omega(s^2) \leq \lambda \omega(s)^2$ for all $s \in S$. Given a nonnegative function $\omega$ on a semigroup $S$, we shall derive necessary and sufficient condition for $\omega$ to have a multiplicativity factor. We shall also do the same for quadrativity factor.

1. Introduction

The paper inspired by the papers [1, 2, 3, 4, 5] of R. Arens, M. Goldburg and W. A. J. Luxemburg. Let $T$ be a normed space seminorm on a complex algebra $A$, i.e., $T(x) \geq 0$, $T(x + y) \leq T(x) + T(y)$ and $T(\alpha x) = |\alpha|T(x)$ for all $x, y \in A$ and $\alpha \in \mathbb{C}$. A seminorm $T$ on an algebra $A$ has a multiplicativity factor (M-factor) [3, 5] if there is $\mu > 0$ such that $T(xy) \leq \mu T(x)T(y)$ for all $x, y \in A$; $T$ has a quadrativity factor (Q-factor) if there is $\lambda > 0$ such that $T(x^2) \leq \lambda T(x)^2$ for all $x \in A$. In [5] they derived necessary and sufficient condition for a seminorm to have a multiplicativity factor. In a subsequent paper [3] they derived necessary and sufficient condition for a seminorm to have a quadrativity factor. Let $\omega$ be a nonnegative function on a semigroup $S$. We shall derive necessary and sufficient for a nonnegative function on a semigroup to have a multiplicativity factor as well as a quadrativity factor.

2. Multiplicative factors

Definition 2.1. Let $S$ be a semigroup, and let $\omega$ be a nonnegative function on $S$. Then

(i) $\omega$ has a multiplicativity factor (M-factor) if there is $\mu > 0$ such that $\omega(st) \leq \mu \omega(s)\omega(t)$ for all $s, t \in S$.

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(ii) $\omega$ has a quadrativity factor (Q-factor) if there is $\lambda > 0$ such that $\omega(s^2) \leq \lambda \omega(s)^2$ for all $s \in S$.

(iii) $\omega$ is a semiweight if $\omega(st) \leq \omega(s)\omega(t)$ for all $s \in S$.

(iv) $\omega$ is a weight if $\omega$ is a semiweight and $\omega(s) > 0$ for all $s \in S$.

It follows from above definition that if $\omega$ has a multiplicativity factor, then it has a quadrativity factor. Also, if $\omega$ is a weight, then $\omega$ is a semiweight. A semiweight $\omega$ on a semigroup $S$ is proper if it not identically zero and $\omega(s) = 0$ for some $s \in S$.

**Definition 2.2.** Let $S$ be a semigroup. A subset $I$ is a semigroup ideal if either $I = \emptyset$ or $(IS \cup SI) \subset I$.

**Definition 2.3.** Let $S$ be a semigroup, and let $\omega : S \to [0, \infty)$ be a map. Then the set $\{s \in S : \omega(s) = 0\}$ is the kernel of $\omega$ and it is denoted by $\ker \omega$.

The following characterizes seminorms having multiplicativity factor [5].

**Theorem 2.4.** [5, Theorem 2.4] Let $A$ be an algebra, and let $T \neq 0$ be a seminorm on $A$. Then

(i) $T$ has a multiplicativity factor if and only if $K = \ker T$ is an ideal in $A$ and

$$
\mu_{\inf} = \sup \{T(xy) : x, y \in A, T(x) = 1 = T(y)\} < \infty.
$$

(ii) If $T$ has a multiplicativity factor and $\mu_{\inf} > 0$, then $\mu$ is a multiplicativity factor if and only if $\mu \geq \mu_{\inf}$.

(iii) If $T$ has a multiplicativity factor and $\mu_{\inf} = 0$, then $\mu$ is a multiplicativity factor if and only if $\mu > 0$.

We have the following analogous result.

**Theorem 2.5.** Let $S$ be a semigroup, and let $\omega : S \to [0, \infty)$ be a map. Then the following statements hold.

(i) $\omega$ has a multiplicativity factor if and only if $\ker \omega$ is an ideal in $S$ and

$$
\mu_{\inf} = \sup \left\{ \frac{\omega(st)}{\omega(s)\omega(t)} : x, y \in S, \omega(x) \neq 0 \neq \omega(y) \right\} < \infty.
$$

(ii) Suppose that $\omega$ has a multiplicativity factor. Then a constant $\mu > 0$ is a multiplicativity factor for $\omega$ if and only if $\mu \geq \mu_{\inf}$. 
Proof. (i) Assume that \( \omega \) has a multiplicativity factor. Then there is \( k > 0 \) such that \( \omega(st) \leq k\omega(s)\omega(t) \) for all \( s, t \in S \). So, if \( s, t \in S, \omega(s) \neq 0 \) and \( \omega(t) \neq 0 \), then \( \frac{\omega(st)}{\omega(s)\omega(t)} \leq k \). Therefore
\[
\sup \left\{ \frac{\omega(st)}{\omega(s)\omega(t)} : s, t \in S, \omega(s) \neq 0, \omega(t) \neq 0 \right\} < \infty.
\]
If \( s \in S \) and \( t \in \ker \omega \), then \( \omega(st) \leq k\omega(s)\omega(t) \) implies that \( \omega(st) = 0 \), i.e., \( st \in \ker \omega \). Also, \( ts \in \ker \omega \). Therefore \( \ker \omega \) is an ideal in \( S \).

Conversely, assume that
\[
\mu_{\inf} = \sup \left\{ \frac{\omega(st)}{\omega(s)\omega(t)} : s, t \in S, \omega(s) \neq 0, \omega(t) \neq 0 \right\} < \infty
\]
and \( \ker \omega \) is an ideal in \( S \). Let \( s, t \in S \). If any of \( \omega(s) \) or \( \omega(t) \) is zero, then \( \omega(st) = 0 \) as \( \ker \omega \) is an ideal in \( S \). This gives \( \omega(st) \leq \mu_{\inf}\omega(s)\omega(t) \). Let \( \omega(s) \neq 0 \) and \( \omega(t) \neq 0 \). Then \( \frac{\omega(st)}{\omega(s)\omega(t)} \leq \mu_{\inf} \). Therefore \( \omega(st) \leq \mu_{\inf}\omega(s)\omega(t) \) for all \( s, t \in S \). So, \( \omega \) has a multiplicativity factor.

(ii) Let \( \mu \) be a multiplicativity factor for \( \omega \). Then \( \frac{\omega(st)}{\omega(s)\omega(t)} \leq \mu \) whenever \( s, t \in S, \omega(s) \neq 0 \) and \( \omega(t) \neq 0 \). But then \( \mu \geq \mu_{\inf} \).

Conversely, assume that \( \mu \geq \mu_{\inf} \). Let \( s, t \in S \). If any of \( \omega(s) \) and \( \omega(t) \) is zero, then \( \omega(st) = 0 \) as \( \ker \omega \) is an ideal in \( S \). So, \( \omega(st) \leq \mu\omega(s)\omega(t) \) in this case. Let \( \omega(s) \neq 0 \) and \( \omega(t) \neq 0 \). Then \( \omega(st) \leq \mu_{\inf}\omega(s)\omega(t) \leq \mu\omega(s)\omega(t) \). Thus \( \mu \) is a multiplicativity factor for \( \omega \).

\[\square\]

Corollary 2.6. Let \( S \) be a semigroup, and let \( \omega \) be a nonzero function on \( S \) such that \( \omega(s) \geq 0 \) for all \( s \in S \). Then \( \omega \) has a multiplicativity factor and 
\[
\mu_{\inf} = \sup \left\{ \frac{\omega(st)}{\omega(s)\omega(t)} : s, t \in S, \omega(s) \neq 0, \omega(t) \neq 0 \right\} = 0 \text{ if and only if } st \in \ker \omega \text{ for all } s, t \in S.
\]

Corollary 2.7. Let \( S \) be a semigroup, and let \( \omega \) be a positive function on \( S \). Then the following statements hold.

(i) \( \omega \) has a multiplicativity factor if and only if \( \mu_{\inf} = \sup \left\{ \frac{\omega(st)}{\omega(s)\omega(t)} : s, t \in S \right\} < \infty \).

(ii) Suppose that \( \omega \) has a multiplicativity factor. Then a real number \( \mu \) is a multiplicativity factor for \( \omega \) if and only if \( \mu \geq \mu_{\inf} \).

Corollary 2.8. If \( S \) is a finite semigroup, and if \( \omega \) is positive function on \( S \), then \( \omega \) has a multiplicativity factor.

Proof. Since \( S \) is a finite set, 
\[
\mu_{\inf} = \sup \left\{ \frac{\omega(st)}{\omega(s)\omega(t)} : s, t \in S \right\} < \infty.
\]
Therefore \( \omega \) has a multiplicativity factor.

\[\square\]
Lemma 2.9. Let $S$ be a semigroup, and let $\omega$ be a nonnegative function on $S$. Suppose that $\omega$ has a multiplicativity factor. Then $\ker \omega$ is an ideal in $S$. If $S$ is a topological semigroup and if $\omega$ is continuous, then $\ker \omega$ is a closed ideal in $S$.

Recall that a semigroup is simple if it has no nontrivial proper ideals.

Corollary 2.10.

(i) If $S$ is a simple semigroup, then there are no multiplicative proper semiweights on $S$.

(ii) If $S$ is a topological semigroup that has no proper closed ideals, then there are no continuous proper semiweights on $S$.

We recall the Rees quotient of a semigroup $S$ by a semigroup ideal $I$. The relation $\sim$ in $S$, defined by $s \sim t$ if either $s = t$ or both $s$ and $t$ are in $I$, is an equivalence relation in $S$. The equivalence classes under $\sim$ are the singleton sets $\{s\}$ with $s \in S \setminus I$ and the set $I$. Since $I$ is an ideal is $S$, the relation $\sim$ is a congruence on $S$. The quotient semigroup $S/I$ is the Rees factor semigroup of $S$ modulo $I$ [6].

Proposition 2.11. Let $\omega$ be a nonnegative function on a semigroup $S$ such that $\ker \omega$ is an ideal in $S$. Define $\bar{\omega} : S/\ker \omega \to [0, \infty)$ by $\bar{\omega}([s]) = \omega(s)$ if $s \in S \setminus \ker \omega$ and $\bar{\omega}(\ker \omega) = 0$. Then $\omega$ has a multiplicativity factor if and only if $\bar{\omega}$ has a multiplicativity factor. In this case, $\mu_{\inf} = \bar{\mu}_{\inf}$.

Proof. Let $\mu > 0$ be a multiplicative factor for $\omega$. Let $[s], [t] \in S/\ker \omega$. If any of $[s]$ and $[t]$ is $\ker \omega$, then $[st] = \ker \omega$ and hence both $\bar{\omega}([s][t])$ and $\bar{\omega}([s])\bar{\omega}([t])$ are zero. Now assume that none of $[s]$ and $[t]$ is $\ker \omega$. Then $\bar{\omega}([s][t]) = \bar{\omega}([st]) = \omega(st) \leq \mu \omega(s)\omega(t) = \mu \bar{\omega}([s])\bar{\omega}([t])$. Therefore $\mu$ is a multiplicativity factor for $\bar{\omega}$.

Let $\mu$ be a multiplicativity factor for $\bar{\omega}$. Let $s, t \in S$. If any of $s$ and $t$ is in $\ker \omega$, then $\omega(st) = 0$ and $\omega(s)\omega(t) = 0$. So, $\omega(st) \leq \mu \omega(s)\omega(t)$. Now assume that none of $s$ and $t$ is in $\ker \omega$. If $st \in \ker \omega$, then clearly $\omega(st) \leq \mu \omega(s)\omega(t)$ holds. Let $st \notin \ker \omega$. Then $\omega(st) = \bar{\omega}([st]) \leq \mu \bar{\omega}([s])\bar{\omega}([t]) = \mu \omega(s)\omega(t)$. Therefore $\omega$ has a multiplicativity factor.

It follows from above that $\mu$ is a multiplicativity factor for $\omega$ if and only if $\mu$ is a multiplicativity factor for $\bar{\omega}$. Thus $\mu_{\inf} = \bar{\mu}_{\inf}$. \qed

The following is analogous to [5, Theorem 1.2].

Theorem 2.12. Let $\omega_1$ and $\omega_2$ be nonnegative functions on $S$, and let $\omega_2$ be submultiplicative, i.e., $\omega_2(st) \leq \omega_2(s)\omega_2(t)$ for all $s, t \in S$. Let $\tau \geq \sigma > 0$ be constants such that

$$\sigma \omega_2(s) \leq \omega_1(s) \leq \tau \omega_2(s) \quad (s \in S).$$
If $\mu \geq \frac{\tau}{\sigma^2}$, then $\mu$ is a multiplicativity factor for $\omega_1$.

Proof. Let $\mu$ satisfy $\mu \geq \frac{\tau}{\sigma^2}$. Let $s, t \in S$. Then

$$\begin{align*}
\mu \omega_1(st) & \leq \mu \tau \omega_2(st) \\
& \leq \mu \tau \omega_2(s) \omega_2(t) \\
& \leq \mu \frac{\tau}{\sigma^2} \omega_1(s) \omega_1(t).
\end{align*}$$

Therefore $\omega_1$ has a multiplicativity factor. \hfill \Box

Corollary 2.13. Let $\omega_1$ and $\omega_2$ be nonnegative functions on a semigroup $S$, and let $\omega_1$ and $\omega_2$ be equivalent, i.e., there are positive constants $\sigma$ and $\tau$ such that $\tau \omega_1 \leq \omega_2 \leq \sigma \omega_1$. Then $\omega_1$ has a multiplicative factor if and only if $\omega_2$ has a multiplicative factor.

Let $T$ be a seminorm on an algebra $A$, and let $T$ have a multiplicativity factor. Given $c \in A$, define $T_c(x) = T(cx)$ for all $x \in A$. It is proved in [5, Theorem 3.1] that $T_c$ has a multiplicativity factor if certain conditions are satisfied. We have a similar result in our case.

Theorem 2.14. Let $S$ be a semigroup, $c \in S$, and let $\omega$ have multiplicative factors. Define $\omega_c(s) = \omega(cs)$ for all $s \in S$. Then $\omega_c$ has a multiplicative factor if any of the following conditions is satisfied.

(i) $S$ has unit and $c$ is invertible.

(ii) $c$ is in the center of $S$ and $c = c^2d$ for some $d \in S$.

Proof. (i) Let $s, t \in S$. Then

$$\begin{align*}
\omega_c(st) & = \omega(cst) \\
& = \omega(cs c^{-1} ct) \\
& \leq \mu \omega(cs) \omega(c^{-1} ct) \\
& \leq \mu^2 \omega_c(s) \omega(c^{-1}) \omega_c(t) \\
& = \mu^2 \omega(c^{-1}) \omega_c(s) \omega_c(t).
\end{align*}$$

Therefore $\omega_c$ has a multiplicative factor.

(ii) Let $s, t \in S$. Then

$$\begin{align*}
\omega_c(st) & = \omega(cst) \\
& = \omega(c^2 dst)
\end{align*}$$
\[ \omega(d \text{csct}) \leq \mu^2 \omega(d) \omega(s) \omega(t) \]

Therefore \( \omega_c \) has a multiplicativity factor. \( \square \)

3. Quadrarive factors

We recall that a nonnegative map \( \omega \) on a semigroup \( S \) has a quadrative factor if there is \( \lambda > 0 \) such that \( \omega(s^2) \leq \lambda \omega(s)^2 \) for all \( s \in S \). The following theorem characterizes seminorms having quadrativity factor [3].

**Theorem 3.1.** [3, Theorem 1.2] Let \( T \) be a seminorm on an algebra \( A \). Then

(i) \( T \) has a quadrativity factor if and only if \( K = \ker T \) is closed under squaring (i.e., \( x^2 \in K \) if \( x \in K \)) and

\[ \lambda_\text{inf} = \sup \{ T(x^2) : x \in A, T(x) \leq 1 \} < \infty. \]

(ii) If \( T \) has a quadravity factor and \( \lambda_\text{inf} > 0 \), then \( \lambda_\text{inf} \) is the best (least) quadravity factor for \( T \).

(iii) If \( T \) has a quadravity factor and \( \lambda_\text{inf} = 0 \), then \( \lambda \) is a quadrativity factor if and only if \( \lambda > 0 \).

The following is a similar result.

**Theorem 3.2.** Let \( S \) be a semigroup, and let \( \omega \) be a nonnegative function on \( S \). Then the following statements hold.

(i) \( \omega \) has a \( Q \)-factor if and only if \( \ker \omega \) is closed under squaring, i.e., \( s^2 \in \ker \omega \) whenever \( s \in S \), and \( \lambda_\text{inf} = \sup \{ \frac{\omega(s^2)}{\omega(s)^2} : s \in S, \omega(s) \neq 0 \} < \infty. \)

(ii) Assume that \( \omega \) has a \( Q \)-factor and \( \lambda_\text{inf} > 0 \). Then \( \lambda > 0 \) is a \( Q \)-factor for \( \omega \) if and only if \( \lambda \geq \lambda_\text{inf} \).

(iii) Assume that \( \omega \) has a \( Q \)-factor and \( \lambda_\text{inf} = 0 \). Then \( \lambda \) is a \( Q \)-factor if and only if \( \lambda > 0 \).

**Proof.** (i) Assume that \( \omega \) has a \( Q \)-factor. Then there is \( \lambda > 0 \) such that \( \omega(s^2) \leq \lambda \omega(s)^2 \) for all \( s \in S \). Let \( s \in \ker \omega \). Then \( \omega(s^2) \leq \lambda \omega(s)^2 = 0 \), i.e., \( s^2 \in \ker \omega \). Therefore \( \ker \omega \) is closed under squaring. If \( s \in S \setminus \ker \omega \), then \( \frac{\omega(s^2)}{\omega(s)^2} \leq \lambda \). Therefore \( \lambda_\text{inf} < \infty \).

Conversely, assume that \( \ker \omega \) is closed under squaring and \( \lambda_\text{inf} < \infty \). Take any \( \lambda > \lambda_\text{inf} \). If \( s \in S \setminus \ker \omega \), then \( \omega(s^2) \leq \lambda_\text{inf} \omega(s)^2 \leq \lambda \omega(s)^2 \). If \( s \in \ker \omega \), then clearly \( \omega(s^2) \leq \lambda \omega(s)^2 \) as \( \ker \omega \).
is closed under squaring. Thus $\omega$ has a $Q$-factor.

(ii) Let $\lambda \geq \lambda_{\text{inf}}$. By (i), $\ker \omega$ is closed under squaring. So, if $s \in \ker \omega$, then $\omega(s^2) \leq \lambda \omega(s)^2$. Let $s \in S \setminus \ker \omega$. Then $\omega(s^2) \leq \lambda_{\text{inf}} \omega(s)^2 \leq \lambda \omega(s)^2$. Thus $\lambda$ is a $Q$-factor for $\omega$.

Conversely, assume that $\lambda$ is a $Q$-factor for $\omega$. Then $\frac{\omega(s^2)}{\omega(s)^2} \leq \lambda$ for all $s \in S \setminus \ker \omega$. Therefore $\lambda_{\text{inf}} \leq \lambda$.

(iii) follows from (ii) \hfill \Box

**Lemma 3.3.** Let $S$ be a semigroup, let $c$ be in the centre of $S$ and $c = c^2$. If a nonnegative function $\omega$ on $S$ has a $Q$-factor, then the map $\omega_c(s) = \omega(cs)$ ($s \in S$) has a $Q$-factor.

**Proof.** Let $\lambda > 0$ be a $Q$-factor for $\omega$. Let $s \in S$. Then

$$\omega_c(s^2) = \omega(cs^2) = \omega(c^2 s^2) = \omega((cs)^2) \leq \lambda \omega(cs)^2 = \lambda \omega_c(s)^2.$$  

Therefore $\omega_c$ has a $Q$-factor. \hfill \Box

**Lemma 3.4.** Let $\omega_1$ and $\omega_2$ be nonnegative functions on $S$, and let $\omega_2(s^2) \leq \omega_2(s)^2$ for all $s \in S$. Let $\tau \geq \sigma > 0$ be constants such that

$$\sigma \omega_2(s) \leq \omega_1(s) \leq \tau \omega_2(s) \quad (s \in S).$$

If $\lambda \geq \frac{\tau}{\sigma}$, then $\lambda$ is a quadrativity factor for $\omega_1$.

**Proof.** Let $\lambda$ satisfy $\lambda \geq \frac{\tau}{\sigma}$. Let $s \in S$. Then

$$\lambda \omega_1(s^2) \leq \lambda \tau \omega_2(s^2) \leq \lambda \tau \omega_2(s)^2 \leq \lambda \frac{\tau}{\sigma} \omega_1(s)^2.$$

This gives $\omega_1(s^2) \leq \frac{\tau}{\sigma} \omega_1(s)^2$. Therefore $\omega_1$ has a $Q$ factor. \hfill \Box

**Corollary 3.5.** Let $\omega_1$ and $\omega_2$ be nonnegative functions on a semigroup $S$, and let $\omega_1$ and $\omega_2$ be equivalent. Then $\omega_1$ has a $Q$-factor if and only if $\omega_2$ has a $Q$-factor.

**Proof.** Since $\omega_1$ and $\omega_2$ are equivalent, there are positive constants $\sigma$ and $\tau$ such that $\tau \omega_1(s) \leq \omega_2(s) \leq \sigma \omega_1(s)$ for all $s \in S$. Assume that $\omega_1$ has a $Q$-factor, say, $\lambda$. Let $s \in S$. Then $\omega_2(s^2) \leq \sigma \omega_1(s^2) \leq \sigma \lambda \omega_1(s)^2 \leq \frac{\tau}{\sigma} \omega_2(s)^2$. Therefore $\omega_2$ has a $Q$-factor. If $\omega_2$ has a $Q$-factor, then it follows from above arguments that $\omega_1$ has a $Q$-factor. \hfill \Box
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