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HOLOMORPHIC ANALOGUES OF CLASSICAL THEOREMS OF WIENER, LEVY AND DOMAR IN FOURIER SERIES

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ABSTRACT. Taylor series and Laurent series analogues for holomorphic functions of classical theorems of Wiener and Levy on absolute convergence of Fourier series as well as their weighted versions on weighted convergence giving analogues of Domar's generalization of Wiener-Levy theorems are obtained.

1. Introduction

The present paper is at the interactions among Complex Analysis, Fourier Series and Banach Algebras. We aim to discuss holomorphic analogues of theorems in Fourier Series due to Wiener [9] and Levy [6]; as well as of their generalizations by Domar [4]. Let f be an integrable function defined on the unit circle Γ. The Fourier series of f is

$$f \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{int}, \text{ where } \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})e^{-int}dt.$$

Let f be continuous on Γ having absolutely convergent Fourier series. The Wiener's theorem states that if $f(z) \neq 0$ for all $z \in \Gamma$, then $\frac{1}{f}$ has absolutely convergent Fourier series. The Levy's theorem states that if F is analytic on an open set containing the range of f, then $F \circ f$ also has absolutely convergent Fourier series. A *weight* on a semigroup (or a group) S is a function $\omega : S \longrightarrow (0, \infty)$ such that $\omega(m+n) \leq \omega(m)\omega(n)$ for all $m, n \in S$. Let ω be a Beurling - Domar weight on $\mathbb{Z}$, i.e., $\omega(n) \geq 1$ $(n \in \mathbb{Z})$ and $\sum\limits_{n\in\mathbb{Z}} \frac{\log\omega(n)}{1+n^2} < \infty$ ([8], p. 185). The Domar's Theorem ([8], Theorem 6.3.2, p. 185) states that if the Fourier series of f is ω - absolutely convergent, i.e., $\sum\limits_{n\in\mathbb{Z}} |\hat{f}(n)|\omega(n) < \infty$, and if $f(z) \neq 0$ for all $z \in \Gamma$, then the Fourier series of $\frac{1}{f}$ is also ω - absolutely convergent.

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Theorem 2.1 below was discussed in [2],which gives holomorphic version of theorems of Wiener and Levy on absolute convergence of Taylor series; whereas Theorem 2.2 gives a weighted version of Theorem 2.1 providing Taylor series analogue of Domar's Theorem. Theorem 3.1 and Corollary 3.2 give Laurent series analogues of Theorems 2.2 and 2.1 respectively. The proof follows Banach algebra arguments (originally due to Gel'fand) using the Beurling algebras $l^1(\mathbb{Z}^+, \omega)$ and $l^1(\mathbb{Z}, \omega)$ in the present case.

2. Weighted Absolute Convergence of Taylor Series

The following was discussed in [2]. We give the proof since it is instructive for the proofs of the remaining results.

Theorem 2.1. *Let f be a complex valued function defined and continuous on the closed unit disc D in the complex plane which is analytic in the interior U of D. Suppose that the Maclaurin series of f is absolutely convergent on Γ. If $f(z) \neq 0$ for all $z \in D$, then the Maclaurin series of $\frac{1}{f}$ is also absolutely convergent on Γ. If F is a function analytic on an open set containing the range of f, then $F \circ f$ also has absolutely convergent Maclaurin series.*

Proof. Let $\mathbb{Z}^+ = \mathbb{N} \bigcup \{0\}$, an additive semigroup. Let $A^+(D)$ consist of complex valued functions f defined and continuous on D, analytic in U and whose Maclaurin series $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ is absolutely convergent. Then $A^+(D)$ is a commutative Banach algebra with pointwise operations and with the norm $\|f\|_1 = \sum\limits_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!}$. In fact, $A^+(D)$ is a Banach algebra of power series having power series generator z, by an example of [3]. Then, as in the proof of Theorem 3.7 of [3], $l^1(\mathbb{Z}^+) \simeq A^+(D)$. We give a proof here. Let $a = (a_n) \in l^1(\mathbb{Z}^+)$. Define $f_a(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. Since $\sum\limits_{n=0}^{\infty} |a_n| < \infty$, $f_a(z)$ is defined for all $z \in D$ and the series $f_a(z)$ is uniformly convergent on U. Thus $f_a(z)$ is analytic in U, continuous on D and $a_n = \frac{f_a^{(n)}(0)}{n!}$, $n \in \mathbb{Z}^+$ by termwise differentiation. Thus $f_a \in A^+(D)$ and $f_a(z) = \sum\limits_{n=0}^{\infty} \frac{f_a^{(n)}(0)}{n!} z^n$ $(z \in D)$. This defines a map $\Lambda : l^1(\mathbb{Z}^+) \longrightarrow A^+(D)$, $\Lambda(a) = f_a$. Clearly Λ is linear and isometric for respective norms $\|\Lambda(a)\|_1 = \|f_a\|_1 = \sum\limits_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!} = \sum\limits_{n=0}^{\infty} |a_n| = \|a\|_1$ $(a \in l^1(\mathbb{Z}^+))$. Further Λ is a homomorphism. Indeed, let $a = (a_n), b = (b_n) \in l^1(\mathbb{Z}^+)$. Then $(a * b)(n) = \sum\limits_{k=0}^{n} a(k)b(n-k)$, $(n \in \mathbb{Z}^+)$. Further

$$\Lambda(a * b)(z) = \sum_{n=0}^{\infty} (a * b)(n) z^n$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(\sum_{k=0}^{n} a(k)b(n-k))z^n \\
&= \sum_{n=0}^{\infty}(\sum_{k=0}^{n} \frac{f_a^{(k)}(0)}{k!}\frac{f_b^{(n-k)}(0)}{(n-k)!})z^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{f_a^{(k)}(0)}{k!}z^k\frac{f_b^{(n-k)}(0)}{(n-k)!}z^{(n-k)} \\
&= (\sum_{n=0}^{\infty}\frac{f_a^{(n)}(0)}{n!}z^n)(\sum_{n=0}^{\infty}\frac{f_b^{(n)}(0)}{n!}z^n) \\
&= f_a(z)f_b(z) \\
&= (f_a.f_b)(z) \\
&= (\Lambda(a)\Lambda(b))(z).
\end{aligned}
$$

Thus $\Lambda(a*b) = \Lambda(a)\Lambda(b)$. We show that Λ is onto. Let $g \in A^+(D)$. Then $\sum\limits_{n=0}^{\infty} \frac{|g^{(n)}(0)|}{n!} < \infty$. For each $n \in \mathbb{Z}^+$, let $a_n = \frac{g^{(n)}(0)}{n!}$. Then $a = (a_n) \in l^1(\mathbb{Z}^+)$ and $\Lambda(a) = g$. Thus $l^1(\mathbb{Z}^+) \simeq A^+(D)$ and so the respective Gel'fand spaces $\Delta(l^1(\mathbb{Z}^+))$ and $\Delta(A^+(D))$ are identified as topological spaces.

Now observe that $\Delta(l^1(\mathbb{Z}^+)) \simeq D$. Indeed, any $z \in D$ defines a complex homomorphism φ_z on $l^1(\mathbb{Z}^+)$ as $\varphi_z(a) = \sum\limits_{n=0}^{\infty} a_n z^n$. Conversely, let $\varphi : l^1(\mathbb{Z}^+) \longrightarrow \mathbb{C}$ be a complex homomorphism. Now, for each k, let δ_k be $\delta_k(n) = 0$ if $n \neq k$, $\delta_k(n) = 1$ if $n = k$. Then $\delta_k \in l^1(\mathbb{Z}^+)$. Let $z = \varphi(\delta_1)$. Then $|z| = |\varphi(\delta_1)| \leq \|\varphi\|\|\delta_1\|_1 = 1$, and $z \in D$. For any $a = (a_n) \in l^1(\mathbb{Z}^+), \varphi(a) = \varphi(\sum\limits_{n=0}^{\infty} a_n\delta_n) = \sum\limits_{n=0}^{\infty} a_n\varphi(\delta_n) = \sum\limits_{n=0}^{\infty} a_n\varphi(\delta_1)^n = \sum\limits_{n=0}^{\infty} a_n z^n = \varphi_z(a)$. Hence $\varphi = \varphi_z$ and $\Delta(l^1(\mathbb{Z}^+)) \simeq D \simeq \Delta(A^+(D))$.

Let $f \in A^+(D)$. Suppose $f(z) \neq 0$ for all $z \in D$. Then $\varphi_z(f) \neq 0$, i.e., $\varphi(f) \neq 0$ for all $\varphi \in \Delta(A^+(D))$. Hence, by the Gel'fand theory, f is invertible in $A^+(D)$, so $\frac{1}{f} \in A^+(D)$. Thus $\frac{1}{f}$ has absolutely convergent Maclaurin series. By the holomorphic functional calculus of $A^+(D)$ [5], $F \circ f \in A^+(D)$; and so $F \circ f$ has absolutely convergent Maclaurin series. This completes the proof. □

Let $\omega : \mathbb{Z}^+ \to (0, \infty)$ be a weight on $\mathbb{Z}^+$. Let $r = \lim\limits_{n\to\infty} \omega(n)^{\frac{1}{n}} > 0$. We say that a series $\sum\limits_{n=0}^{\infty} a_n$ of complex numbers is *ω-absolutely convergent* if $\sum\limits_{n=0}^{\infty} |a_n|\omega(n) < \infty$. Let $D_r = \{z \in \mathbb{C} : |z| \leq r\}$. We have the following weighted analogue of above result.

Theorem 2.2. *Let f be a complex valued function defined and continuous on the closed disc D_r in the complex plane which is analytic in the interior U_r of D_r. Suppose that f has ω-absolutely convergent Maclaurin series. If $f(z) \neq 0$ for all $z \in D_r$, then $\frac{1}{f}$ also* *has ω-absolutely convergent Maclaurin series. Further if F is analytic on an open set containing range of f, then $F \circ f$ also has ω-absolutely convergent Maclaurin series.*

Proof. Consider the algebra $A_\omega(D_r)$ of functions defined and continuous on D_r, analytic on U_r, and for which $\sum\limits_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!}\omega(n) < \infty$. It is a Banach algebra with pointwise operations and with the norm $\|f\|_\omega = \sum\limits_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!}\omega(n)$. As in the previous result, $A_\omega(D_r)$ is isometrically isomorphic to the Beurling Banach algebra $l^1(\mathbb{Z}^+, \omega)$ consisting of scalar sequences $a = (a_n)$ such that $\sum\limits_{n=0}^{\infty} |a_n|\omega(n) < \infty$ and having the norm $\|a\|_{1,\omega} = \sum\limits_{n=0}^{\infty} |a_n|\omega(n)$(See also examples 1.2, 1.3 and Theorem 3.7 of [3]). Again, as in the previous result, $\Delta(A_\omega(D_r)) \simeq \Delta(l^1(\mathbb{Z}^+, \omega)) \simeq D_r$.

Now let f be as in the statement of the theorem. Then $f \in A_\omega(D_r)$ and $\varphi(f) \neq 0$ for all $\varphi \in \Delta(A_\omega(D_r))$. Hence, by the Gel'fand theory, f is invertible in $A_\omega(D_r)$ and $\frac{1}{f} \in A_\omega(D_r)$. Thus $\frac{1}{f}$ has ω-absolutely convergent Maclaurin series. By the holomorphic functional calculus of $A(D_r)$ in Banach algebras [5], $F \circ f \in A_\omega(D_r)$ as spectrum of f in $A_\omega(D_r)$ being the range of f. This completes the proof. $\square$

Remark 2.3. Let H_0^ω consists of (equivalence classes of) functions defined and analytic on neighbourhoods of 0 such that $\sum\limits_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!}\omega(n) < \infty$. Such a function f is, in fact, defined on D_r and analytic on $U_r = \{z \in \mathbb{C} : |z| < r\}$. Indeed, the series $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ is convergent and defined for all z such that $|z^n| \leq \omega(n)$ for all n, i.e., for all z such that $|z| \leq \lim\limits_{n \longrightarrow \infty} \omega(n)^{\frac{1}{n}} = r$. The series is uniformly convergent on U_r, so that $f(z) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ is defined on D_r and analytic on U_r. Thus H_0^ω coincides with the collection $A_\omega(D_r)$ of functions defined and continuous on D_r, analytic on U_r, and for which $\sum\limits_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!}\omega(n) < \infty$. Thus, in above theorem, it is sufficient to assume that f is defined and analytic on some neighbourhood of 0 and it has ω-absolutely convergent Maclaurin series.

3. Weighted Absolute Convergence of Laurent Series

Let $\omega : \mathbb{Z} \longrightarrow (0, \infty)$ be a weight on the additive group $\mathbb{Z}$. Note that we do not assume $\omega \geq 1$. Let $r_+ = \lim\limits_{n \longrightarrow \infty} \omega(n)^{\frac{1}{n}} = \inf\{\omega(n)^{\frac{1}{n}} : n \in \mathbb{N}\}$. Let $r_- = \lim\limits_{n \longrightarrow -\infty} \omega(n)^{\frac{1}{n}} = \lim\limits_{n \longrightarrow \infty} \omega(-n)^{\frac{-1}{n}} = \sup\{\omega(n)^{\frac{1}{n}} : n \in \mathbb{N}^-\}$. Then $0 < r_- \leq r_+ < \infty$. Let $D[r_-, r_+] = \{z \in \mathbb{C} : r_- \leq |z| \leq r_+\}$ be the closed annulus having interior $U(r_-, r_+) = \{z \in \mathbb{C} : r_- < |z| < r_+\}$ which is the open annulus. Let $0 < r_1 < r_2 < \infty$. Let f be a function continuous on $D[r_1, r_2]$ and analytic on $U(r_1, r_2)$. By Laurent's theorem in Complex Analysis (Section 4.12 of [7]), f admits a unique representation $f(z) = \sum\limits_{n \in \mathbb{Z}} \hat{f}(n) z^n$ $(z \in U(r_1, r_2))$, where

$\hat{f}(n) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z^{n+1}}dz$ with $C_r = \{z \in \mathbb{C} : |z| = r\}$ for $r_1 < r < r_2$. The series is uniformly convergent in $D[\rho_1, \rho_2]$ where $r_1 < \rho_1 < \rho_2 < r_2$. We say that f has ω - *absolutely convergent Laurent series* if $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|\omega(n) < \infty$. Note that $\hat{f}(n)$ does not depend on the choice of r. Also, depending on ω, it may happen that $U(r_-, r_+) = \phi$, in which case, $r_- = r_+ = r$. In this case also, the Laurent series $f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)z^n$, $\hat{f}(n) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z^{n+1}}dz$, makes sense. We also need the Banach algebra $A_\omega(D[r_-, r_+])$ consisting of functions $f \in A(D[r_-, r_+])$ having ω - absolutely convergent Laurent series (Example 3 of [1]).

Theorem 3.1. *Let ω be a weight on $\mathbb{Z}$. Let f be a function continuous on $D[r_-, r_+]$ and analytic in $U(r_-, r_+)$. Assume that f has ω - absolutely convergent Laurent series.*

(1) *If $f(z) \neq 0$ for all $z \in D[r_-, r_+]$, then $\frac{1}{f}$ also has ω - absolutely convergent Laurent series.*

(2) *Let F be a function analytic on an open set containing the range of f. Then $F \circ f$ also has ω - absolutely convergent Laurent series.*

Proof. Consider the annulus Banach algebra $A(D[r_-, r_+])$ consisting of functions f continuous on $D[r_-, r_+]$ and analytic in $U(r_-, r_+)$ with pointwise operations and the sup norm $\|f\|_\infty = \sup\{|f(z)| : z \in D[r_-, r_+]\}$. The algebra $A_\omega(D[r_-, r_+])$ is a subalgebra of $A(D[r_-, r_+])$ and is a Banach algebra with norm $\|f\|_\omega = \sum_{n\in\mathbb{Z}} |\hat{f}_n|\omega(n)$. Further notice that $A_\omega(D[r_-, r_+])$ is isometrically isomorphic to the convolution Beurling Banach algebra $l^1(\mathbb{Z}, \omega)$ (see also Lemma 3.5 and main Theorem of [1]). Let $a = (a_n)_{-\infty}^{\infty} \in l^1(\mathbb{Z}, \omega)$. Define a function f_a formally as $f_a(z) = \sum_{n\in\mathbb{Z}} a_n z^n = \sum_{n\geq 0} a_n z^n + \sum_{n<0} a_n z^n = f_{a,1}(z) + f_{a,2}(z)$ (say). By the definition of r_+, the series for $f_{a,1}(z) = \sum_{n\geq 0} a_n z^n$ is uniformly convergent on $U(r_+)$. By the definition of r_-, the series for $f_{a,2}(z) = \sum_{n<0} a_n z^n$ is uniformly convergent on $\{z \in \mathbb{C} : |z| > r_-\}$. Hence $f_a(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ is continuous on $D[r_-, r_+]$ and analytic in $U(r_-, r_+)$. By the uniqueness of the Laurent expansion, $\sum_{n\in\mathbb{Z}} a_n z^n$ is the Laurent series for f_a with $a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f_a(z)}{z^{n+1}}dz$ with $C_r = \{z \in \mathbb{C} : |z| = r\}$ for some r, $r_- \leq r \leq r_+$. Since $\sum_{n\in\mathbb{Z}} |a_n|\omega(n) < \infty$, it follows that $f_a \in A_\omega(D[r_-, r_+])$. This defines a map $\Lambda : l^1(\mathbb{Z}, \omega) \longrightarrow A_\omega(D[r_-, r_+])$. Clearly the map Λ is linear and isometric, since $\|\Lambda(a)\|_\omega = \sum_{n\in\mathbb{Z}} |a_n|\omega(n) = \|a\|_\omega$. Again, by using the definition of convolution of elements $a = (a_n), b = (b_n) \in l^1(\mathbb{Z}, \omega)$, we see that $\Lambda(a * b) = \Lambda(a)\Lambda(b)$ showing that Λ is a homomorphism. The map Λ is also onto. Given $f \in A_\omega(D[r_-, r_+])$ having Laurent series expansion $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, there exists a sequence $a = (a_n) \in l^1(\mathbb{Z}, \omega)$ such that $\Lambda(a) = f$. Thus $l^1(\mathbb{Z}, \omega) \simeq A_\omega(D[r_-, r_+])$. Now by Proposition 2.2.8 of [5], $D[r_-, r_+]$ is homeomorphic to the Gel'fand space $\Delta(l^1(\mathbb{Z}, \omega))$ of $l^1(\mathbb{Z}, \omega)$, the homeomorphism given by $z \longrightarrow \varphi_z$ where $\varphi_z(a) = \sum_{n\in\mathbb{Z}} a_n z^n, \ a \in l^1(\mathbb{Z}, \omega)$.

Now let f be as in statement of the theorem. Then $f \in A_\omega(D[r_-, r_+])$. Let $f(z) \neq 0$ for all $z \in D[r_-, r_+]$. Hence, by above, $\varphi(f) \neq 0$ for all $\varphi \in \Delta(A_\omega(D[r_-, r_+])$. Hence f is invertible in $A_\omega(D[r_-, r_+])$. Thus $\frac{1}{f} \in A_\omega(D[r_-, r_+])$, so that $\frac{1}{f}$ has ω - absolutely convergent Laurent series. This proves (i). For (ii), let F be analytic on an open set containing the range of f. Now the range of f is $\{f(z) : z \in D[r_-, r_+]\} = \{\varphi(f) : \varphi \in \Delta(A_\omega(D[r_-, r_+]))\}$. By the holomorphic functional calculus of $A(D[r_-, r_+])$ in Banach algebras ([5], Section 3.1), $F(f) \in A_\omega(D[r_-, r_+])$ and $F \circ f$ has ω - absolutely convergent Laurent series. This completes the proof. $\square$

The following can be regarded as the Laurent series analogues of Theorems of Wiener and Levy. We say that a function f holomorphic on $U(r_1, r_2)$ has *absolutely convergent Laurent series* if the Laurent series expansion of f satisfies $\sum_{n\geq 0} |\hat{f}(n)| r_2^n + \sum_{n<0} |\hat{f}(n)| r_1^n < \infty$.

Corollary 3.2. *Let f be a complex valued function defined and continuous on the closed annulus $D[r_1, r_2]$ in the complex plane which is analytic in the interior $U(r_1, r_2)$ of $D[r_1, r_2]$. Suppose that Laurent series of f is absolutely convergent in $D[r_1, r_2]$.*

(1) *If $f(z) \neq 0$ for all $z \in D[r_1, r_2]$, then $\frac{1}{f}$ also has absolutely convergent Laurent series in $D[r_1, r_2]$.*
(2) *Let F be an analytic function on the range of f. Then $F \circ f$ also has absolutely convergent Laurent series in $D[r_1, r_2]$.*

This follows from the previous Theorem by taking weight ω on $\mathbb{Z}$ as $\omega(n) = r_1^n \ (n < 0), \ \omega(n) = r_2^n \ (n \geq 0)$.

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