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Generalized fractional kinetic equations involving generalized extended τ- hypergeometric Function

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Abstract

Recently, introduced an extension of generalized τ-hypergeometric function ${}_3\Gamma_2^{\tau}(z)$ (R. K. Gupta et al. [5]),[6]. In the present paper, authers have established further generalization of fractional kinetic equations involving generalized extended τ-hypergeometric functions. The solution of these generalized fractional kinetic equations were obtained in term of Mittag -Leffler function using Laplace transform.

Keywords: Generalized extended incomplete τ-hypergeometric Function, Fractional kinetic equation, Laplace transforms, Mittag-Leffler function, Fractional calculus.

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1 Introduction and Preliminaries

1.1 Generalized extended τ-hypergeometric Function

Extended τ hypergeometric function ${}_3\Gamma_2^{\tau}(z)$ was given by [5], [6] and defined as follows:

$$\begin{aligned} {}_3\Gamma_2^{\tau}(z) &= {}_3\Gamma_2^{\tau}((\lambda,k),a,b;c,d;z) \\ &= \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{[\lambda;k]_n\,\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)} \frac{z^n}{n!}, \end{aligned} \tag{1.1}$$

$$(k \geq 0; \tau > 0; |z| < 1, \Re(d) > \Re(a) > 0,\, \Re(c) > \Re(b) > 0 \,\text{when}\, k = 0).$$

Notes:-

(i) If we put b = d, then (1.1) reduces to the extended τ-hypergeometric function ${}_2\Gamma_1^{\tau}(z)$ given by Parmer([10] p.422,eq.(2.1)) as defined as

$${}_2\Gamma_1^{\tau}(z) = {}_2\Gamma_1^{\tau}((\lambda,k),b;c;z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{[\lambda;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \tag{1.2}$$

$$(\lambda, b \in \mathbb{C}, c \in \mathbb{C} \setminus \mathbb{Z}_0^-, k \geq 0; \tau > 0; |z| < 1, \Re(c) > \Re(b) > 0 \, when \, k = 0).$$

(ii) If we put b = d and $\tau = 1$, then (1.1) reduces to the extended Gauss hypergeometric function([3] p. 487, eq.17) given by

$$_2F_1(z) = \, _2F_1((\lambda, k), b; c; z) = \sum_{n=0}^{\infty} \frac{[\lambda; k]_n \, (b)_n}{(c)n} \frac{z^n}{n!}. \tag{1.3}$$

(iii)If we take $\tau = 1$ and $k = 0$ in (1.1), then it reduces to the classical Gauss's hypergeometric function as

$$_3\Gamma_2(z) = \, _3\Gamma_2((\lambda, a, b; c, d; z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n \, (a)_n \, (b)_n}{(c)_n \, (d)_n} \frac{z^n}{n!} \tag{1.4}$$

(iv)If we take $b = d$, $\tau = 1$ and $k = 0$ in (1.1), then it reduces to the classical Gauss's hypergeometric function as

$$_2F_1(z) = \, _2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n \, (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{1.5}$$

1.2 Fractional Kinetic Equations-

If an arbitrary reaction is characterized by a time dependent $N = N(t)$ then it is possible to calculate the rate of change of $\frac{dN}{dt}$ by mathematical equation

$$\frac{dN}{dt} = -d + p,$$

where d is the destruction rate and p is the production rate of N.

Haubold and Mathai [2] established a functional differential equation between the rate of change of reaction, the destruction rate and the production rate as follow:

$$\frac{dN}{dt} = -d(N_t) + p(N_t), \tag{1.6}$$

where $N = N(t)$ is the rate of reaction, $d(N(t))$ is the rate of destruction, $p(N_t)$ is the rate of production and N_t denotes the function defined by $N_t(t^*) = N(t) - t^*, t^* > 0$.

A special case of (1.6), when spatial fluctuations or homogeneities in the quantity $N(t)$, are neglected, given by following differential equation (Haubold and Mathai [2] and Kourganoff [19]):

$$\frac{dN_i}{dt} = -c_i N_i(t), \tag{1.7}$$

where initial condition $N_t(t = 0) = N_0$ is the number of density of species i at time $t = 0, c_i > 0$.

Solution of standard kinetic equation (1.7) is given by (Kourganoff [19]) as

$$N_i(t) = N_0 e^{-c_i t}.$$

If we decline the index i and integrate standard kinetic equation (1.7), we have

$$N(t) - N_0 = -c_0 \, {}_0D_t{}^{-1} N(t),$$

where ${}_0D_t{}^{-1}$ is standard integral operator.

Haubold and Mathai [2] obtained the fractional generalization of the standard kinetic equation (1.7) as

$$N(t) - N_0 = -c_0^{\nu} \, {}^{RL}{}_{0+}I_t{}^{\nu} N(t), \tag{1.8}$$

where ${}^{RL}{}_{0+}I_t{}^{\nu} = {}_0D_t{}^{-\nu}$ is Riemann-Liouville fractional integral operator defined as [16].

$${}^{RL}{}_{0+}I_t{}^{\nu}\{f(t)\} = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} du, t > 0, \Re(\nu) > 0. \tag{1.9}$$

Laplace transform of a function $f(z)$(Sneddon [7]) is given by

$$L\{f(z) : s\} = \int_0^{\infty} e^{-sz} f(z) dz. \tag{1.10}$$

The laplace transform (1.10) of the Riemann- Liouville fractional integral operatoris given by ([16])

$$L\{{}^{RL}{}_{0+}I_t{}^{\nu} f(t); s\} = s^{-\nu} L\{f(t); s\}. \tag{1.11}$$

Solution of equation (1.8) is given by (Haubold and Mathai [2])

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k + 1)} (c_0 t)^{\nu k}.$$

Two dimensional Mittag-Leffler function [9] is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(\alpha k + \beta)}; \quad \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0. \tag{1.12}$$

2 Generalized Fractional Kinetic Equations

In this section, generalized fractional kinetic equation established as theorems and their special cases (Corollaries).

Theorem 1. *If $d > 0,\ p > 0,\ k \geq 0,\ \tau > 0,\ t > 0; \Re(q) > \Re(u) > 0; \Re(w) > \Re(v) > 0$ when $k = 0$ and $|\frac{d^p}{s^p}| < 1$, then the solution of equation*

$$N(t) - N_0 \, {}_3\Gamma_2^{\tau}(t) = -d^p \, {}^{RL}{}_{0+}I_t{}^{p} N(t) \tag{2.1}$$

is by

$$N(t) = N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda, k]_n \, \Gamma(u + \tau n)\, \Gamma(v + \tau n)}{\Gamma(w + \tau n)\, \Gamma(q + \tau n)} \, t^n \, E_{p,n+1}\left(-d^p \, t^p\right)$$

where $E_{\alpha,\beta}(x)$ is the generalized Mittag-Leffler function given by (1.12).

Proof:- Applying Laplace transform (1.10) on (2.1) and using (1.11), we have

$$L\{N(t);s\} = N_0 L\{{}_3\Gamma_2^{\tau}(t);s\} - d^p L\{{}^{RL}{}_{0+}I_t{}^p N(t);s\}$$

$$L\{N(t);s\} = N_0 \left(\int_0^{\infty} e^{-st} \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda,k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)} \frac{t^n}{n!} dt \right) - d^p s^{-p} L\{N(t);s\}$$

therefore,

$$(1+d^p s^{-p}) L\{N(t);s\} = N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda,k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)\,n!} \int_0^{\infty} e^{-st} t^n\,dt$$

this gives,

$$L\{N(t);s\} = N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda,k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)} \frac{1}{s^{n+1}} \frac{1}{(1+d^p s^{-p})}$$

after simplification of above equation, we get

$$L\{N(t);s\} = N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda,k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)} \left\{ \sum_{r=0}^{\infty} (-1)^r\, d^{pr} s^{-(pr+n+1)} \right\} \tag{2.2}$$

taking inverse Laplace transform of (2.2) and using $L^{-1}\{s^{-p};t\} = \frac{t^{p-1}}{\Gamma(p)}$, we arrived at

$$\begin{aligned} N(t) &= N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda,k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)} L^{-1} \left\{ \sum_{r=0}^{\infty} (-1)^r\, d^{pr} s^{-(pr+n+1)} \right\} \\ &= N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda,k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)} \left\{ \sum_{r=0}^{\infty} (-1)^r d^{pr} \frac{t^{pr+n}}{\Gamma(pr++n+1)} \right\} \\ &= N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda,k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)} t^n \left\{ \sum_{r=0}^{\infty} \frac{(-d^p\,t^p)^r}{\Gamma(pr+n+1)} \right\} \end{aligned}$$

hence,

$$N(t) = N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda,k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)} t^n\, E_{p,n+1}(-d^p\,t^p)$$

Further, if we put $v = q$ in Theorem 1 reduces to corollary as the extended τ-hypergeometric function.

Corollary 1. *If* $d > 0,\ p > 0,\ k \geq 0,\ \tau > 0,\ t > 0; \Re(w) > \Re(v) > 0$ *and* $|\frac{d^p}{s^p}| < 1$*, then the solution of equation*

$$N(t) - N_0\ {}_2\Gamma_1^{\tau}(t) = -d^p\ {}^{RL}{}_{0+}I_t{}^p N(t) \tag{2.3}$$

is by

$$N(t) = N_0 \frac{\Gamma(w)}{\Gamma(v)} \sum_{n=0}^{\infty} \frac{[u,k]_n\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)} t^n\, E_{p,n+1}(-d^p\,t^p)$$

where $E_{\alpha,\beta}(x)$ *is the generalized Mittag-Leffler function given by* (1.12).

If we put $v = q, \tau = 1$ and $k = 0$ in Theorem 1 reduces to corollary as the Classical Gauss's hypergeometric function.

Theorem 2. *If* $d > 0,\ p > 0,\ k \geq 0,\ \tau > 0,\ t > 0; \Re(q) > \Re(u) > 0; \Re(w) > \Re(v) > 0$ *when* $k = 0$ *and* $|\frac{d^p}{s^p}| < 1$*, then the solution of equation*

$$N(t) - N_0\ {}_3\Gamma_2^{\tau}(d^p\, t^p) = -d^p\ {}^{RL}{}_{0+}I_t{}^{p} N(t) \tag{2.4}$$

is given by,

$$N(t) = N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda, k]_n\, \Gamma(u + \tau n)\, \Gamma(v + \tau n)}{\Gamma(w + \tau n)\, \Gamma(q + \tau n)} \frac{d^n\, t^n}{n!}\, \Gamma(pn + 1)\, E_{p,pn+1}\,(-d^p\, t^p)$$

Proof:- Applying Laplace transform (1.10) on (2.4) and using (1.11), we get

$$L\{N(t); s\} = N_0 L\{{}_3\Gamma_2^{\tau}(d^p t^p); s\} - d^p L\{{}^{RL}{}_{0+}I_t{}^{p} N(t); s\}$$

$$\begin{aligned} L\{N(t); s\} =& N_0 \left(\int_0^{\infty} e^{-st} \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda, k]_n\, \Gamma(u + \tau n)\, \Gamma(v + \tau n)}{\Gamma(w + \tau n)\, \Gamma(q + \tau n)} \frac{d^{pn}\, t^{pn}}{n!} dt \right) \\ & - d^p s^{-p} L\{N(t); s\} \end{aligned}$$

therefore,

$$\begin{aligned} (1 + d^p s^{-p}) L\{N(t); s\} =& N_0 \left(\sum_{k=0}^{\infty} \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda, k]_n\, \Gamma(u + \tau n)\, \Gamma(v + \tau n)}{\Gamma(w + \tau n)\, \Gamma(q + \tau n)} \frac{d^{pn}}{n!} \right) \\ & \int_0^{\infty} e^{-st} t^{pn}\, dt \end{aligned}$$

i.e.

$$L\{N(t); s\} = N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda, k]_n\, \Gamma(u + \tau n)\, \Gamma(v + \tau n)}{\Gamma(w + \tau n)\, \Gamma(q + \tau n)} \frac{d^{pn}}{n!} \frac{\Gamma(pn + 1)}{s^{pn+1}} \frac{1}{(1 + d^p s^{-p})}$$

above equation can be written as

$$\begin{aligned} L\{N(t); s\} =& N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda, k]_n\, \Gamma(u + \tau n)\, \Gamma(v + \tau n)}{\Gamma(w + \tau n)\, \Gamma(q + \tau n)} \frac{d^{pn}}{n!} \\ & \Gamma(pn + 1) \left\{ \sum_{r=0}^{\infty} (-1)^r\, d^{pr}\, s^{-(pr+pn+1)} \right\} \end{aligned} \tag{2.5}$$

taking inverse Laplace transform of (2.5), we get

$$\begin{aligned} N(t) =& N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda, k]_n\, \Gamma(u + \tau n)\, \Gamma(v + \tau n)}{\Gamma(w + \tau n)\, \Gamma(q + \tau n)} \frac{d^{pn}}{n!} \Gamma(pn + 1) \\ & L^{-1} \left\{ \sum_{r=0}^{\infty} (-1)^r\, d^{pr}\, s^{-(pr+pn+1)} \right\} \end{aligned}$$

using $L^{-1}\{s^{-p};t\} = \frac{t^{p-1}}{\Gamma(p)}$, we obtain

$$N(t) = N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda, k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)}\,\frac{d^{pn}}{n!}\,\Gamma(pn+1) \left\{\sum_{r=0}^{\infty} (-1)^r\, d^{pr} \frac{t^{pr+pn}}{\Gamma(pr+pn+1)}\right\}$$

finally,

$$N(t) = N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda, k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)}\,\frac{(d\,t)^{pn}}{n!}\,\Gamma(pn+1) \times \left\{\sum_{r=0}^{\infty} \frac{(-d^p\,t^p)^r}{\Gamma(pr+pn+1)}\right\}$$

hence,

$$N(t) = N_0 \frac{\Gamma(w)\,\Gamma(q)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{[\lambda, k]_n\,\Gamma(u+\tau n)\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,\Gamma(q+\tau n)}\,\frac{(d\,t)^{pn}}{n!}\Gamma(pn+1) E_{p,pn+1}(-d^p\,t^p)$$

Further, if we put $v = q$ in Theorem 2 reduces to corollary as the extended τ-hypergeometric function.

Corollary 2. *If* $d > 0,\ p > 0,\ k \geq 0,\ \tau > 0,\ t > 0; \Re(w) > \Re(v) > 0$ *and* $|\frac{d^p}{s^p}| < 1$*, then the solution of equation*

$$N(t) - N_0\ {}_2\Gamma_1^{\tau}(d^p\,t^p) = -d^p\ {}^{RL}{}_{0+}I_t{}^p N(t) \tag{2.6}$$

is by

$$N(t) = N_0 \frac{\Gamma(w)}{\Gamma(v)} \sum_{n=0}^{\infty} \frac{[u, k]_n\,\Gamma(v+\tau n)}{\Gamma(w+\tau n)\,n!}\,\Gamma(pn+1) E_{p,pn+1}(-d^p\,t^p)$$

where $E_{\alpha,\beta}(x)$ *is the generalized Mittag-Leffler function given by* (1.12).

If we put $v = q, \tau = 1$ and $k = 0$ in Theorem 2 reduces to corollary as the Classical Gauss's hypergeometric function.

3 Conclusion

In the present paper authers investigated solution of generalized fractional kinetic equation in term of extended τ-hypergeometric functions and some special cases.

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