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Commutative rings with finite hyper-graph and automorphisms of maximal hyper-graphs on $\mathbb{Z}_n$

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Abstract

Let R be a commutative ring with identity. For any integer $k > 1$, an element is a k-zero divisor if there are k distinct elements including the given one, such that the product of all is zero but the product of fewer than all is nonzero. Let $Z(R, k)$ denote the set of the k-zero divisors of R. A ring with no k-zero divisors is called a k-domain. In this paper we claim the following results:

Theorem 3.2: Suppose R has a finite but non empty $Z(R, k)$. Then for any $a \in R$, either a is a unit or a zero divisor. Further, the set of units of R is finite and the nil-radical N is finite.

Proposition 3.4: Suppose R is a k- domain for some $k \geq 2$. Then R is is an l-domain for all $l \geq k$. In other words, $HG(R) < k$.

Theorem 4.1: Group of automorphisms $Aut(H_k(\mathbb{Z}_n))$ is a product of symmetric groups, where H_k is the maximal hyper-graph on $\mathbb{Z}_n$. In other words,

$$\mathrm{Aut}(H_k(\mathbb{Z}_n)) = \prod_1^m S_{\phi_i}$$

We also give explicit formula to describe a maximal hyper-graph and degree of arbitrary vertex.

Keywords: Hyper-graph, commutative rings, ideals, k-zero divisors, total ring of fractions.

AMS Subject Classification(2010): 13A05, 13E99, 13F15, 5C25

1 Introduction

A simple graph is an ordered pair (V, E), where V is a vertex set and E is an edge set with edges of the form $\{v_1, v_2\}$ where v_1, v_2 are two distinct elements in V. A zero divisor graph on a commutative ring R is a simple graph $\Gamma(R)$ whose vertex set is the set of zero divisors $Z(R)$.

Set $\{v, w\}$ form an edge if $v \cdot w = 0$. The study of a zero divisor graph on a commutative ring was first introduced in 1988 by Beck in [**5**]. Anderson and Livingston [**2**] modified this definition by removing zero from the vertex set.

For various results on zero divisor graphs, we suggest [**1**], [**2**], [**3**] and [**7**]. Hyper-graphs generalizes the concept of graphs. A hyper-graph is a pair (V, E) of a vertex set V and an edge set E. But unlike graphs, an edge may contain any number of vertices (see Berge [**6**] for detailed definition). In 2007, zero divisor graphs were generalized to k-zero divisor hyper-graphs in [**8**]. Zero divisors were replaced by k-zero divisors defined here in the definition 2.1. In this paper, we verify some finiteness necessary conditions of finite hyper-graph rings. In section 2 we record definitions and introduce the notations used in this paper. In section 3 we study commutative rings with finite or null hyper-graphs. In section 4 we consider maximal hyper-graphs $H_k(\mathbb{Z}_n,)$ over $\mathbb{Z}_n$ and classify the group of automorphisms $\mathrm{Aut}(H_k(\mathbb{Z}_n))$.

2 Definitions and notations

All rings considered in this paper are commutative with identity. Most terms from the theory of commutative rings are taken from [**4**]. An element a in a ring R is called a **zero divisor** if there is a nonzero element b such that $a \cdot b = 0$. We will denote the set of zero divisors by $Z(R)$. An element a is called nilpotent if there exist an exponent m such that $a^m = 0$. The set of all the nilpotent elements forms an ideal which is called a **nil-radical** and is denoted by $N(R)$ or just N. **Annihilator** of an element $a \in R$ is an ideal defined as $\mathrm{Ann}(a) = \{x \in R \mid x \cdot a = 0\}$. We recall the following definition from [**8**].

Definition 2.1. Let R be a commutative ring and $k \geq 2$ be a fixed integer. Element $v_1 \in R$ is called a k-**zero divisor** if there exist $v_2, v_3, \ldots, v_k$ in R such that (1) $\{v_1, v_2, \ldots, v_k\}$ are all distinct elements, (2) $\Pi_1^k v_i = 0$ and (3) $\Pi_{i \neq j} v_i \neq 0$ for any j, $1 \leq j \leq k$.

The set of all k-zero divisors is denoted by $Z(R, k)$. Clearly $Z(R, k) \subset Z(R)$. A ring in which $Z(R, k)$ is a null set is called a k-**integral domain** or just a k-**domain**. A ring for which $Z(R, k)$ is finite but nonempty is called a **finite hyper-graph ring** or just a **finite HG ring**.

We will now define a k uniform hyper-graph $H_k(R)$ on R. The vertex set is $Z(R, k)$. Elements $v_1, v_2, \ldots, v_k$ which appear in the definition 2.1 form a k-**edge** (edge with k elements) of the hyper-graph. We give some examples. All examples considered in this section are taken from [**10**] and [**11**] where they are presented with detailed description.

Example 2.1. We consider the ring $\mathbb{Z}_{12} = \{0, 1, \ldots, 11\}$ with operations addition and multiplication modulo 12. The 3-uniform hyper-graph $H_3(\mathbb{Z}_{12})$ has the vertex set $Z(\mathbb{Z}_{12}, 3) =$

$\{2,3,9,10\}$. There are two 3-edges, $\{2,3,10\}$ and $\{2,9,10\}$ which are represented by the two ovals in figure 1.

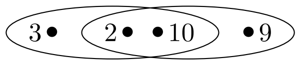

Figure 1: Three uniform hyper-graph on $\mathbb{Z}_{12}$.

Note that $\mathbb{Z}_{12}$ is a 4- domain.

The concept of hyper-graphic constant is introduced in **[11]**, we restate it in the following definition.

Definition 2.2. For any ring R, we define a **hyper-graphic constant** $HG(R)$ to be the smallest integer g such that R is a k-domain for all $k > g$. For any domain, we define $HG(R) = 0$. If R is not a domain but is a k-domain for all $k \geq 2$, then we say that $HG(R) = 1$. If no such integer exists, then we say that $HG(R) = \infty$.

Note that in the definition 2.2, R is not a g-domain, that is, $Z(R, g) \neq \emptyset$. If $HG(R) = g < \infty$, then the hyper-graph $H_g(R)$ will be referred to as the **maximal** hyper-graph of R .

Example 2.2. Since $\mathbb{Z}$ is a domain, $HG(\mathbb{Z}) = 0$. $HG(\mathbb{Z}_{12}) = 3$ and $HG(\mathbb{Z}_4) = 1$. In fact, $HG(R) = 1$ if and only if R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/ < x^2 >$.

Example 2.3. $H_3(\mathbb{Z}_{18})$ has vertex set

$$Z(\mathbb{Z}_{18}, 3) = \{2, 3, 4, 8, 10, 14, 15, 16\}$$

and six 3-edges, {2,3,15},{4,3,15},{8,3,15}, {10,3,15},{14,3,15} and {16,3,15}, as shown in Figure 2.

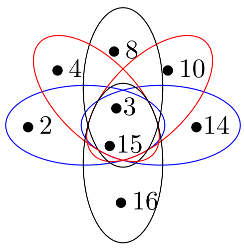

Figure 2: Three uniform hyper-graph $H_3(\mathbb{Z}_{18})$.

Note that $\mathbb{Z}_{18}$ is a k-domain for any $k > 3$, hence, $HG(\mathbb{Z}_{18}) = 3$.

Example 2.4. Let K be a field and $K[x] = K[x_1, x_2, \ldots,]$ be a polynomial ring in infinitely many variables. Let I be an ideal generated by the monomials x_i^i. Then, the quotient ring $A = K[x]/I$ is not a k-domain for any k. Hence $HG(A) = \infty$.

Example 2.5. Suppose R is a direct product of n copies of domains, then $HG(R) = n$.

We end this section with a note on units in a ring R. The set of units, denoted by $U(R)$, forms a group under multiplication. If $R = \mathbb{Z}_n$, then $U(R)$ is simply denoted by $U(n)$. Cardinality of $U(n)$ can be computed using Euler ϕ function (see page 107 in [**12**]). When $n = \prod_1^m p_i^{\alpha_i}$,

$$\phi(n) = n\prod_i^m \left(1 - \frac{1}{p_i}\right) = \prod_1^m p_i^{\alpha_i - 1}(p_i - 1).$$

Further, if n is a composite integer larger than 8,, then by the algorithm developped in [**11**], $HG(\mathbb{Z}_n) = \sum \alpha_i$. Moreover, we can describe the maximal hyper-graph $H_k(\mathbb{Z}_n)$ as follows:

- Vertex set $Z(\mathbb{Z}_n, k) = \dot{\bigcup}_1^m U_i$, where $U_i = p_i \cdot U(n/p_i)$.
- Cardinality $|U_i| = \phi_i = \phi(n/p_i)$ and $|Z(R,k)| = \sum_1^m \phi_i$.
- Any k edge Λ contains exactly α_i elements from the set U_i, which are of the form $p_i \cdot u$ where u is a unit modulo n/p_i.
- Number of edges $= |E| = \Pi_{i=1}^m \binom{\phi_i}{\alpha_i}$.
- If $v = p_j \cdot u$ for some unit u, then the number of edges containing v is

$$d(v) = \binom{\phi_j - 1}{\alpha_j - 1} \cdot \prod_{i \neq j} \binom{\phi_i}{\alpha_i}.$$

we apply this algorithm to find maximal hyper-graph on $\mathbb{Z}_{18}$.

Example 2.6. Suppose $n = 18 = 2 \cdot 3^2$. Then $\alpha_1 = 1$, $\alpha_2 = 2$ and $HG(\mathbb{Z}_{18}) = 1 + 2 = 3$ as noted in example 2.3. Further,

$$U_1 = 2 \cdot U(9) = 2 \cdot \{1, 2, 4, 5, 7, 8\} = \{2, 4, 8, 10, 14, 16\}$$

$$U_2 = 3 \cdot U(6) = 3 \cdot \{1, 5\} = \{3, 15\}$$

Hence, $Z(\mathbb{Z}_{18}, 3) = U_1 \cup U_2 = \{2, 4, 8, 10, 14, 16, 3, 15\}$

Also since $\phi_1 = 3 \cdot 2 = 6$, $\phi_2 = 2$, $|Z(\mathbb{Z}_{18}, 3)| = 8$

and $|E| = \binom{6}{1} \cdot \binom{2}{2} = 6$

which agrees with our previous record.

Example 2.7. Suppose $n = 2^2 \cdot 5^2 = 100$. Then $k = 4$ and $H_4(\mathbb{Z}_{100})$ is the maximal hyper-graph on $\mathbb{Z}_{100}$. Now since $\phi_1 = 5 \cdot 4 = 20$ and $\phi_2 = 2 \cdot 4 = 8$, $H_4(\mathbb{Z}_{100})$ has $\phi_1 + \phi_2 = 28$ vertices and $\binom{20}{2} \cdot \binom{8}{2} = 5,320$ edges. Further $d(2) = \binom{19}{1} \cdot \binom{8}{2} = 512$.

When $HG(R) = 1$, then R is completely determined. It is interesting question to determine rings by their hyper-graphic constant.

3 Finite hyper-graph rings

In this section we will study finite HG rings. In theorem 3.2, we will obtain some necessary finiteness conditions. First we prove a lemma. In principle, the technique used in the lemma is present in the proposition 5.3 in **[11]**, we merely isolate the idea and present it here.

Lemma 3.1. *Let R be a ring with finite and nonempty $Z(R,k)$. Then for any k edge, $\Lambda = \{v_1, v_2, \ldots, v_k\}$ and for all j, $1 \le j \le k$, the ideal $\cap_{i \ne j} \operatorname{Ann}(v_i)$ is finite.*

Proof. Without loss of generality, we can assume that $j = 1$. Suppose $I = \cap_{i \ne 1} \operatorname{Ann}(v_i)$ is an infinite ideal. First note that for any distinct pair a, $b \in I$, $v_1 + a \ne v_1 + b$. Since I is an infinite set, we can find infinite subset X such that $v_1 + a \notin \Lambda \cup \{0\}$ for any $a \in X$.

Now for any $a \in X$, we define $\Lambda(a) = \{w_1, \ldots, w_k\}$ where, $w_1 = v_1 + a$ and $w_i = v_i$ for $2 \le i \le k$. We will verify three conditions of definition 2.1. By our choice of X, condition (1) holds. Also since $a \in \operatorname{Ann}(v_i)$ for $i \ge 2$,

$$\prod_1^k w_i = \prod_1^k v_i + a \cdot \prod_2^k v_i = \prod_1^k v_i = 0.$$

This implies that condition (2) also hods. For condition (3), we first note that $\prod_2^k w_i = \prod_2^k v_i \ne 0$. If $j \ne 1$, then calculation similar to one used in condition (2) shows that

$$\prod_{i \ne j} w_i = (v_1 + a) \prod_{i \ne 1, j} v_i = \prod_{i \ne j} v_i \ne 0.$$

Thus $\Lambda(a)$ is a k edge for all $a \in X$ and we have infinitely many k zero divisors $\{v_1 + a \mid a \in X\}$. This contradicts the finiteness of $Z(R,k)$. Hence, I must be finite. □

In the following theorem, we will show that for any finite HG ring R, $R = T(R)$, where $T(R)$ is the total ring of fraction. Instead of verifying finiteness of $Z(R,k)$, (which depends on k), it suffices to just look for a nonzero divisor!

Theorem 3.2. *Suppose R is a finite HG ring. Then for any $a \in R$, either a is a unit or a zero divisor. Further, the set of units of R is finite and the nil-radical N is finite.*

Proof. Suppose $a \in R$ is a nonzero divisor which is not a unit. Let $\Lambda = \{v_1, v_2, \ldots, v_k\}$ be a k-edge. Then for any integer $n \ge 1$, $a^n \cdot \Lambda = \{a^n \cdot v_1, \ldots, a^n \cdot v_k\}$ is also a k-edge. Thus $a^n \cdot v_2 \in Z(R,k)$ for all n. Since $Z(R,k)$ is finite, there exist a pair $s < t$ such that $a^s \cdot v_2 = a^t \cdot v_2$ or $v_2 = a^{t-s} \cdot v_2$. Set $n_2 = t - s$. Then $n_2 \ge 1$ and $v_2 = a^{m_2} \cdot v_2$. Since a^{n_2} is also a nonzero divisor, we can replace a^{n_2} by a and assume that $v_2 = a \cdot v_2$. Clearly, for any exponent m, $v_2 = a^m \cdot v_2$.

Now we find $n_3 \geq 1$ such that $v_3 = a^{n_3} \cdot v_3$. Again we replace a^{n_3} by a to get a value a such that $v_i = a^m \cdot v_i$ for all $m \geq 1$ and for $i = 2, 3$. Continuing this way, we can find a nonzero divisor $a \in R$, such that for any exponent m, $v_i = a^m v_i$ for $2 \leq i \leq k$. That is $(1 - a^m) \in \operatorname{Ann}(v_i)$ for all $i > 1$ and,

$$Y = \{1 - a^m \mid m \geq 1\} \subset \cap_2^k \operatorname{Ann}(v_i).$$

Now if for some positive exponents $t > s$, $1 - a^t = 1 - a^s$, then $a^{t-s}(a^s - 1) = 0$. But a is neither a unit nor a zero divisor, therefore, the product can not be zero. This shows that the Y is an infinite set. But $Y \subset \cap_{i \neq 1} \operatorname{Ann}(v_i)$. This contradicts the lemma 3.1. Therefore, no such a exist and the first assertion is proved.

Suppose $U(R)$ is infinite. Let $a_0 \in U(R)$. Then $a_0 \cdot \Lambda$ is a k-edge. Hence $a_0 \cdot v_2$ is a k-zero divisor. Since there are infinitely many units and only finitely many k-zero divisors, we can find infinite subset $A_2 = \{a_1, a_2, \ldots\}$ of $U(R)$ such that for all i, $a_i \cdot v_2 = a_0 \cdot v_2$. We now replace $U(R)$ with A_2 and find an infinite subset $A_3 = \{b_1, b_2, \ldots\}$ of A_2 such that for all $b_i \in A_2$, $b_0 \cdot v_j = b_i \cdot v_j$ for $j = 2$, 3 for some $b_0 \in A_2$.

Continuing this way, we can find a unit c_0 and an infinite subset $A_k = \{c_1, c_2, \ldots\}$ of $U(R)$ such that $c_i \cdot v_j = c_0 \cdot v_j$ for all $2 \leq j \leq k$. That is $c_i - c_o \in \operatorname{Ann}(v_j)$ for all i and j or

$$Y = \{c_i - c_0 \mid i \geq 1\} \subset \cap_2^k \operatorname{Ann}(v_j),$$

Clearly Y is an infinite subset, which contradicts lemma 3.1. Therefore, $U(R)$ must be a finite set.

For the last assertion, we only need to note that for any nil-potent element a, $1 - a$ is a unit. Since $U(R)$ is finite, so is N. This completes the proof. □

As we noted before, the above theorem shows that nonempty and finite $Z(R, k)$ implies $R = T(R)$, Now for any ring S, $S \subset T(S)$. Therefore, $Z(S, k)$ is a subset of $Z(T(S), k)$. This shows that the converse of theorem 3.2 is not true, for example, take $R = T(S)$, where the ring S has infinitely many k zero divisors.

We end this section with a result on rings with no k zero divisors, namely, k domains. It is a natural question to ask, if R is a k domain, is it also a $k + 1$ domain? Before attempting this question, we consider the following proposition.

Proposition 3.3. *Suppose R is a ring which is not a k- domain for some $k \geq 3$,. Then R is not a $k - 1$ domain.*

Proof. Let $\Lambda = \{v_1, v_2, \ldots, v_k\}$ be a k-edge. If for some $j < l$, $v_j v_l \notin \Lambda$, then consider

$$\Lambda_1 = \{v_1, \ldots, v_{j-1}, v_j \cdot v_l, v_{j+1} \ldots, v_{l-1}, v_{l+1}, \ldots, v_k\}.$$

By our assumption, Λ_1 contains distinct elements. Conditions (2) and (3) of the definition 2.1 are easy to verify. Therefore Λ_1 is a $k-1$ edge which proves our assertion.

Alternatively, suppose that for all pairs $i < j$, $v_i \cdot v_j \in \Lambda$. If for some pair, $v_i \cdot v_j = v_i$, then we can substitute this relation in $\Pi_{l \neq j} v_l$ to get

$$\begin{aligned}\Pi_{l \neq j} v_l &= v_1 \cdot v_2 \ldots v_{j-1} \cdot v_{j+1} \ldots v_k \\ &= v_1 \ldots v_{i-1} \cdot (v_i \cdot v_j) \cdot v_{i+1} \ldots v_{j-1} \cdot v_{j+1} \ldots v_k \\ &= \Pi_{l=1}^{k} v_l = 0.\end{aligned}$$

Since Λ is a k-edge, this can not be true. Therefore, $v_i \cdot v_j \neq v_i$. Similarly, we can prove that $v_i \cdot v_j \neq v_j$.

Thus $v_1 \cdot v_2 = v_h$ for some $h > 2$. By reordering Λ, if required, we can assume that $v_1 \cdot v_2 = v_3$. Next, we consider $v_2 \cdot v_3$. If $v_2 \cdot v_3 = v_1$, then since $v_3 = v_1 \cdot v_2$,

$$\Pi_{l \neq 2} v_l = v_1 \cdot v_3 \ldots v_k = (v_2 \cdot v_3) \cdot (v_1 \cdot v_2) \cdot v_4 \ldots v_k = 0$$

which is a contradiction. Therefore, $v_2 \cdot v_3 \neq v_1$ Also by our previous argument, $v_2 \cdot v_3$ is not equal to v_2 or v_3. Hence $v_2 \cdot v_3 = v_h$ for some $h > 3$. Once again we reorder Λ and assume that $v_2 \cdot v_3 = v_4$. We continue this process until we find indexes i, j, $3 \leq i < j < k$ such that $v_{j-2} \cdot v_{j-1} = v_i$. We may assume that j is the smallest index with this property. That is, for each $l < j$, $v_{l-2} \cdot v_{i-1} = v_l$. If $j = i+1$ or $i+2$, then $j-1 = i$ or $j-2 = i$, which can be rulled out by our previous argument. Thus $j > i+2$. We claim that $\Pi_{l \neq i-1} v_l = 0$.

$$\begin{aligned}\Pi_{l \neq i-1} v_l &= v_1 \ldots v_{i-2} \cdot v_i \ldots v_j \ldots v_k \\ &= v_1 \ldots v_{i-2} \cdot v_i \cdot \underline{(v_{i-1} \cdot v_i)} \ldots (v_{j-3} \cdot v_{j-2}) v_j \ldots v_k\end{aligned}$$

Now the underlined factor has one copy of v_i which we replace with the relation $v_i = v_{j-2} \cdot v_{j-1}$. This way we have all v_i present in the product. Therefore $\Pi_{l \neq i-1} v_l = 0$. This is a contradiction. Thus our assumption that for all pairs $i < j$, $v_i \cdot v_j \in \Lambda$ is false. This completes the proof. □

An easy induction proves the following corollary.

Corollary. *Suppose R is a ring which is not a k- domain for some $k \geq 3$. Then R is not a j domain for any j, $2 \leq j \leq k$.*

Contra-positive of the above result is quite interesting.

Proposition 3.4. *Suppose R is a k- domain for some $k \geq 2$. Then R is is an l-domain for all $l \geq k$. In other words, $HG(R) < k$.*

4 $\mathrm{Aut}(H_k(\mathbb{Z}_n))$ of the maximal hyper-graphs

Let R be a commutative ring and $k \geq 2$ be an integer. Suppose $\Lambda = \{v_1, \ldots, v_k\} \in H_k(R)$ and $a \in R$ is a unit. Then

$$a \cdot \Lambda = \{a \cdot v_1, \ldots, a \cdot v_k\}$$

is also a k-edge. Thus

$$\rho_a : Z(R, k) \to Z(R, k),\ \rho_a(v) = a \cdot v.$$

is a well defined edge preserving bijection. Such functions are called an **automorphisms** of a hyper-graph. In the following definition, we have restricted ourselves to k uniform hyper-graphs, but it can serve for any hyper-graph with minor modifications.

Definition 4.1. A one to one and onto function F defined on $Z(R, k)$ to $Z(R, k)$ is called an automorphism of a hyper-graph $H_k(R)$ if for any k-edge $\Lambda = \{v_1, \ldots, v_k\}$, $F(\Lambda) = \{F(v_1) \ldots, F(v_k)\}$ is also a k-edge. The set of all the automorphisms forms a group and is denoted by $\mathrm{Aut}(H_k(R))$.

If F is any ring automorphism, then restriction of F to $Z(R, k)$ give arise to an automorphism of $H_k(R)$.

Example 4.1. Let $n = 2^3$. Then $HG(\mathbb{Z}_8) = 2$ (and not 3 as we noted at the end of section 2). Hence, $H_2(\mathbb{Z}_8)$ is the maximal hyper-graph. The vertex set of $H_2(\mathbb{Z}_8)$ is $Z(\mathbb{Z}_8, 2) = \{2, 4, 6\}$. We can write $Z(\mathbb{Z}_8, 2) = V_1 \cup V_2$ where $V_1 = \{2, 6\}$ and $V_2 = \{4\}$. Then any automorphism of $H_2(\mathbb{Z}_8)$ fixes V_1 and V_2. Hence,

$$\mathrm{Aut}(H_2(\mathbb{Z}_8)) \simeq S(V_1) \times S(V_2) \simeq \mathbb{Z}_2.$$

Suppose $n = \prod_1^m p_i^{\alpha_i}$ is a composite integer larger than 8, then the maximal hyper-graph on $\mathbb{Z}_n$ is $H_k(\mathbb{Z}_n)$ where $k = \sum \alpha_i$ (see section 6 in **[11]**). In the following theorem we show that the group $\mathrm{Aut}(H_k(\mathbb{Z}_n))$ is isomorphic to a direct product of symmetric groups.

Theorem 4.1. *Suppose* $n = \prod_1^m p_i^{\alpha_i}$ *and* $k = \sum \alpha_i$ *with* $n > 8$. *Then,* $\mathrm{Aut}(H_k(\mathbb{Z}_n)) = \prod_1^m S_{\phi_i}$ *where* $\phi_j = \phi(n/p_j)$.

Proof. First observe that for any v, $w \in U_i = p_i \cdot U(n/p_i)$, permutation (v, w) is a well defined bijective function on $Z(R, k)$. Clearly it also preserves edge relation. Therefore, $(v,\ w)$ is an automorphism. But transpositions generate $S_{|U_i|} = S_{\phi_i}$, therefore,

$$\prod_1^m S_{\phi_i} \subset \mathrm{Aut}(H_k(\mathbb{Z}_n)).$$

To prove the reverse implication, we will show that each U_i remains invariant under any automorphism F. To the contrary, suppose $F(v_1) = w_1$ where $v_1 \in U_i$, $w_1 \in U_j$ for some

$i \neq j$. Also suppose that $\Lambda = \{v_1, \ldots, v_k\}$ and $F(\Lambda) = \{w_1, \ldots, w_k\}$. Now since $n > 8$, by **[11]** $\phi_i > \alpha_i$. Let $v \in U_i - \Lambda$. Consider $\Lambda_1 = \{v, v_2, \ldots, v_k\}$. Then $F(\Lambda_1) = \{F(v), w_2, \ldots, w_k\}$ is also a k edge. Hence $\prod(F(\Lambda_1)) = 0$. Now in $\{w_2, \ldots, w_k\}$, only $\alpha_j - 1$ elements are from U_j. Therefore, $F(v) \in U_j$. Now assuming that $v_2 \in U_i$, if $F(v_2) \in U_l$ for some l, then using the same argument we can say that $F(v) \in U_l$ which is absurd. Therefore,$F(U_i \cap \Lambda) \subset U_j \cap \Lambda$ or $\alpha_i \leq \alpha_j$. Using similar argument for F^{-1}, we can show that $\alpha_j \leq \alpha_i$ or $\alpha_i = \alpha_l$. Now F preserves degree of vertices. Therefore, $d(v_1) = d(w_1)$. But

$$d(v_1) = \binom{\phi_i - 1}{\alpha_i - 1} \prod_{l=2}^{m} \binom{\phi_l}{\alpha_l}, \; d(w_1) = \binom{\phi_j - 1}{\alpha_j - 1} \prod_{l \neq j}^{m} \binom{\phi_l}{\alpha_l}$$

Hence,

$$\binom{\phi(n/p_i) - 1}{\alpha_i - 1} \prod_{2}^{m} \binom{\phi(n/p_l)}{\alpha_l} = \binom{\phi(n/p_j) - 1}{\alpha_j - 1} \prod_{l \neq 2}^{m} \binom{\phi(n/p_l)}{\alpha_l}$$

which implies that,

$$\frac{\alpha_i}{\phi(n/p_i)} \prod_{1}^{m} \binom{\phi(n/p_l)}{\alpha_l} = \frac{\alpha_j}{\phi(n/p_j)} \prod_{1}^{m} \binom{\phi(n/p_l)}{\alpha_l}$$

Simplifying both sides gives

$$\frac{\alpha_i}{\phi(n/p_i)} = \frac{\alpha_j}{\phi(n/p_j)} \text{ or } p_i \cdot \alpha_i = p_j \cdot \alpha_j.$$

Since $\alpha_i = \alpha_j$, $p_i = p_j$, which is a contradiction. Therefore U_i is invariant under F and the proof is complete. □

Example 4.2. For $n = 12 = 2^2 \cdot 3$, $H_3(\mathbb{Z}_{12})$ is the maximal hyper-graph of $\mathbb{Z}_{12}$ (see example 2.1 and figure 1 of section 2). The vertex set is $Z(\mathbb{Z}_{12}, 3) = \{2, 3, 9, 10\}$ and the edge set contains $\{2, 3, 10\}$ and $\{2, 9, 10\}$. Any automorphisms F must fix {3, 9} and {2, 10}. Therefore,

$$\text{Aut}(H_3(12)) = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Example 4.3. Suppose $n = 18 = 2 \cdot 3^2$. Then $p_1 = 2$, $\alpha_1 = 1$, $p_2 = 3$, $\alpha_2 = 2$. Therefore, $k = 3$ and $H_3(\mathbb{Z}_{18})$ is maximal. Now $U_1 = 2 \cdot U(9) = \{2, 4, 8, 10, 14, 16\}$ and $\phi_1 = 6$. Similarly $U_2 = 3 \cdot U(6) = \{3, 15\}$ and $\phi_2 = 2$. Therefore,

$$\text{Aut}(H_3(\mathbb{Z}_{18})) = S_6 \times \mathbb{Z}_2.$$

Example 4.4. Suppose $n = 180 = 2^2 \cdot 3^2 \cdot 5$. We will compute ϕ_i.

$$\phi_i = \phi(90) = \phi(2 \cdot 3^2 \cdot 5) = 24$$

$$\phi_2 = \phi(60) = \phi(2^2 \cdot 3 \cdot 5) = 16$$

$$\phi_3 = \phi(36) = \phi(2^2 \cdot 3^2) = 12$$

Hence,

$$\text{Aut}(H_5(\mathbb{Z}_{180})) = S_{24} \times S_{16} \times S_{12}.$$

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