Mathematics Today Vol.35 (December 2019) 44-49
ISSN 0976-3228, e-ISSN 2455-9601

SOME TELESCOPING SERIES FOR k-FIBONACCI AND k-LUCAS SEQUENCES

A. D. GODASE

Abstract. In this paper, we establish some telescoping series for k-Fibonacci and k-Lucas sequences and prove their relationships with k-Fibonacci and k-Lucas sequences. This approach is different for k-Fibonacci sequence literature.

1. Introduction

The Fibonacci sequence is a source of many nice and interesting identities.Many researchers studied and generalised these sequences, different identities of these sequences are appeared in [6], [7], [4], [10], [12], [13], [17]. A similar interpretation exists for k-Fibonacci and k-Lucas numbers. Many of these identities have been documented in the work of Falcon and Plaza [1], [2], [3], where they proved them by algebraic means, many interesting algebraic identities are also proved in [5], [8], [9]. In this paper, we prove some results related with telescoping series of k-Fibonacci and k-Lucas sequences.

2. Preliminary

Definition 2.1. The $k-$Fibonacci sequence $\{F_{k,n}\}_{n=1}^{\infty}$ is defined as, $F_{k,n+1} = k \cdot F_{k,n} + F_{k,n-1}$, with $F_{k,0} = 0$, $F_{k,1} = 1$, for $n \geq 1$.

Definition 2.2. The $k-$ Lucas sequence $\{L_{k,n}\}_{n=1}^{\infty}$ is defined as, $L_{k,n+1} = k \cdot L_{k,n} + L_{k,n-1}$, with $L_{k,0} = 2$, $L_{k,1} = k$, for $n \geq 1$.

Characteristic equation of the initial recurrence relation is

$$r^2 - k \cdot r - 1 = 0. \tag{2.1}$$

Characteristic roots are

$$r_1 = \frac{k + \sqrt{k^2+4}}{2},$$
$$r_2 = \frac{k - \sqrt{k^2+4}}{2}.$$

Date: Thursday 16^{th} January, 2020.

1991 *Mathematics Subject Classification.* Primary 11B39; Secondary11B37.

Key words and phrases. Fibonacci Sequence, k-Fibonacci Sequence, k-Lucas Sequence.

Characteristic roots verify the properties

$$r_1 - r_2 = \sqrt{k^2+4} = \sqrt{\Delta} = \delta,$$
$$r_1 + r_2 = k,$$
$$r_1.r_2 = -1.$$

Binet forms for $F_{k,n}$ and $L_{k,n}$ are

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2},$$
$$L_{k,n} = r_1^n + r_2^n.$$

Using Binet forms following identities can be obtained easily

$$2F_{k,m+n} = F_{k,m}L_{k,n} + L_{k,m}F_{k,n},$$
$$2F_{k,m-n} = (-1)^n(F_{k,m}L_{k,n} - L_{k,m}F_{k,n}).$$

3. Telescoping series for k-Fibonacci and k-Lucas sequences

In this section, we establish some telescoping series for k-Fibonacci and k-Lucas sequences.

Theorem 3.1. *For m, $n \geq 0$, we have*

$$\sum_{i=1}^{i=n}(-1)^{im}\frac{1}{F_{k,(i+1)m}F_{k,im}} = \frac{1}{2F_{k,m}}\sum_{i=1}^{i=n}\left[\frac{L_{k,mi}}{F_{k,mi}} - \frac{L_{k,m(i+1)}}{F_{k,m(i+1)}}\right]$$
$$= \frac{1}{2F_{k,m}}\left[\frac{L_{k,m}}{F_{k,m}} - \frac{L_{k,m(n+1)}}{F_{k,m(n+1)}}\right],$$
$$\sum_{i=0}^{i=n}(-1)^{im}\frac{1}{F_{k,(i+1)m}L_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}},$$
$$\sum_{i=1}^{i=n}(-1)^{im}\frac{F_{k,(2i+1)m}}{F^2_{k,(i+1)m}F^2_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}},$$
$$\sum_{i=0}^{i=n}(-1)^{im}\frac{F_{k,(2i+1)m}}{L^2_{k,(i+1)m}L^2_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}},$$
$$\sum_{i=0}^{i=n}(-1)^{im}\frac{F^3_{k,(2i+1)m} + F_{k,(2i+1)m}F^2_{k,m}}{L^4_{k,(i+1)m}L^4_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}}.$$

Proof. For m, $n \geq 0$, we have

$$\frac{F_{k,(2n+1)m}}{L_{k,(n+1)m}L_{k,nm}} = \frac{1}{2}\left[\frac{F_{k,(n+1)m}}{L_{k,(n+1)m}} + \frac{F_{k,nm}}{L_{k,nm}}\right],$$

$$\frac{F_{k,(2n+1)m}}{F_{k,(n+1)m}F_{k,nm}} = \frac{1}{2}\left[\frac{L_{k,(n+1)m}}{F_{k,(n+1)m}} + \frac{F_{k,nm}}{L_{k,nm}}\right],$$

$$\frac{(-1)^{mn}F_{k,m}}{F_{k,(n+1)m}F_{k,nm}} = \frac{1}{2}\left[\frac{L_{k,nm}}{F_{k,n)m}} - \frac{L_{k,(n+1)m}}{F_{k,(n+1)m}}\right],$$

$$\frac{(-1)^{mn}F_{k,m}}{L_{k,(n+1)m}L_{k,nm}} = \frac{1}{2}\left[\frac{F_{k,(n+1)m}}{L_{k,(n+1)m}} - \frac{F_{k,nm}}{L_{k,nm}}\right],$$

For first sum,

$$\begin{aligned}\sum_{i=1}^{i=n}(-1)^{im}\frac{1}{F_{k,(i+1)m}F_{k,im}} &= \frac{1}{2F_{k,m}}\sum_{i=1}^{i=n}\left[\frac{L_{k,mi}}{F_{k,mi}} - \frac{L_{k,m(i+1)}}{F_{k,m(i+1)}}\right] \\ &= \frac{1}{2F_{k,m}}\left[\frac{L_{k,m}}{F_{k,m}} - \frac{L_{k,m(n+1)}}{F_{k,m(n+1)}}\right],\end{aligned}$$

$$\sum_{i=0}^{i=n}(-1)^{im}\frac{1}{F_{k,(i+1)m}L_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}}.$$

For second sum,

$$(-1)^{mi}\frac{F_{k,(2i+1)m}F_{k,m}}{F^2_{k,(i+1)m}F^2_{k,im}} = \frac{1}{4}\left[\frac{L^2_{k,(i+1)m}}{F^2_{k,(i+1)m}} - \frac{L^2_{k,im}}{F^2_{k,im}}\right],$$

$$\sum_{i=1}^{i=n}(-1)^{im}\frac{F_{k,(2i+1)m}}{F^2_{k,(i+1)m}F^2_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}}.$$

For third sum,

$$(-1)^{mi}\frac{F_{k,(2i+1)m}F_{k,m}}{L^2_{k,(i+1)m}L^2_{k,im}} = \frac{1}{4}\left[\frac{F^2_{k,(i+1)m}}{L^2_{k,(i+1)m}} - \frac{F^2_{k,im}}{L^2_{k,im}}\right].$$

For fourth sum,

$$\sum_{i=0}^{i=n}(-1)^{im}\frac{F_{k,(2i+1)m}}{L^2_{k,(i+1)m}L^2_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}},$$

$$\begin{aligned}\frac{F^2_{k,(2i+1)m}+F^2_{k,m}}{L^2_{k,(i+1)m}L^2_{k,im}} &= \frac{1}{4}\left[(\frac{F^2_{k,(i+1)m}}{L_{k,(i+1)m}} + \frac{F_{k,im}}{L_{k,im}})^2 + (\frac{F_{k,(i+1)m}}{L_{k,(i+1)m}} - \frac{F_{k,im}}{L_{k,im}})^2\right] \\ &= \left[\frac{F^2_{k,(i+1)m}}{L^2_{k,(i+1)m}} + \frac{F^2_{k,im}}{L^2_{k,im}}\right],\end{aligned}$$

$$(-1)^{mi}\frac{F_{k,(2i+1)m}(F^2_{k,m}+F^2_{k,(i+1)m}}{L^4_{k,(i+1)m}L^4_{k,im}} = \frac{1}{8}\left[\frac{F^4_{k,(i+1)m}}{L^4_{k,(i+1)m}} - \frac{F^4_{k,im}}{L^4_{k,im}}\right],$$

$$\sum_{i=0}^{i=n}(-1)^{im}\frac{F^3_{k,(2i+1)m}+F_{k,(2i+1)m}F^2_{k,m}}{L^4_{k,(i+1)m}L^4_{k,im}} = \frac{F_{k,m(n+1)}}{2F_{k,m}L_{k,m(n+1)}}.$$

□

Lemma 3.2. *For variable k and non-negative integer r, we have*

$$\sum_{i\geq 0}\binom{r}{i}(1+k)^i = rk^{r-1}(1+k) \tag{3.1}$$

Lemma 3.3. *For positive integer m, n the solution of the simultaneous equations*

$$1 + xr_1^m = yr_1^{-n},$$
$$1 + xr_2^m = yr_2^{-m}$$

for the unknown x and y is

$$x = -\frac{F_{k,n}}{F_{k,m+n}}, \quad y = (-1)^n\frac{F_{k,m}}{F_{k,m+n}}.$$

Proof. We have

$$r_1^n + xr_2^{m+n} = y = r_2^n + xr_2^{m+n}.$$

Since, $m + n \neq 0$, we get

$$x = \frac{r_2^n - r_1^n}{r_1^{m+n} - r_2^{m+n}} = -\frac{F_{k,n}}{F_{k,m+n}}.$$

Similarly,

$$-r_1^{-m} + yr_2^{-(m+n)} = x = -r_2^{-m} + yr_2^{-(m+n)},$$

$$y = \frac{r_1^{-m} - r_2^{-m}}{r_1^{-(m+n)} - r_2^{-(m+n)}} = \frac{F_{k,-m}}{F_{k,-(m+n)}} = \frac{(-1)^{m+1}F_{k,m}}{(-1)^{m+n+1}F_{k,m+n}} = (-1)^{-n}\frac{F_{k,m}}{F_{k,m+n}}.$$

□

Theorem 3.4. *Let r be a non-negative integer and for positive integers p, q, we have*

$$\sum_{i\geq 0}\binom{r}{i}F_{k,p}^iF_{p+q}^{r-i}L_{k,qi} = (-1)^{q+1}F_{k,p}F_{k,q}^{r-1}L_{k,pr-(p+q)}.$$

Proof. Since $r_1r_2 = -1$,

$$r_1^{-q} = (-1)^q r_2^q,$$
$$x_1 = xr_1^p = -\frac{F_{k,q}}{F_{k,p+q}}r_1^p$$
$$(1 + x_1) = yr_1^{-q} = (-1)^q yr_2^q = \frac{F_{k,p}}{F_{k,p+q}}r_2^q.$$

$$\sum_{i\geq 0}(-1)^{r-i}i\binom{r}{i}\left(\frac{F_{k,p}}{F_{k,p+q}r_2^q}\right)^i = r\left(-\frac{F_{k,q}}{F_{k,p+q}r_1^p}\right)^{r-1}\left(\frac{F_{k,p}}{F_{k,p+q}r_2^q}\right)$$
$$= (-1)^{q+r-1}r\frac{F_{k,p}F_{k,q}^{r-1}}{F_{k,p+q}^r r_1^{pr-(p+q)}}$$

Using lemma (3.3), we get

$$x_1 = -\frac{F_{k,q}}{F_{p+q}}r_2^p,$$
$$(1+x_1) = \frac{F_{k,p}}{F_{p+q}}r_1^q.$$

$$\sum_{i\geq 0}(-1)^{r-i}i\binom{r}{i}\frac{F_{k,p}^i}{F_{p+q}^i}r_1^{qi} = (-1)^{q+r-1}r\frac{F_{k,p}F_{k,q}^{r-1}}{F_{k,p+q}^r}r_2^{pr-(p+q)},$$
$$\sum_{i\geq 0}\binom{r}{i}F_{k,p}^iF_{p+q}^{r-i}L_{k,qi} = (-1)^{q+1}F_{k,p}F_{k,q}^{r-1}L_{k,pr-(p+q)}.$$

□

Lemma 3.5. *For $n \geq 0$, we have*

$$k\sum_{j=0}^{n-1}\binom{2n-1-j}{j}(k^2+4)^{n-j-1}(-1)^j = F_{k,2n}$$

Theorem 3.6. *For $n \geq 0$, we have*

$$\sum_{i=0}^{\left[\frac{n}{2}\right]}\binom{n-i}{i}k^iF_{k,3i} = \frac{kF_{k,2n+1}-F_{k,2n}+(-k)^{n+2}F_{k,n}+(-k)^{n+1}F_{k,n-1}}{(2k^2-1)},$$
$$\sum_{i=0}^{\left[\frac{n}{2}\right]}\binom{n-i}{i}k^iL_{k,3i} = \frac{kL_{k,2n+1}-L_{k,2n}+(-k)^{n+2}L_{k,n}+(-k)^{n+1}L_{k,n-1}}{(2k^2-1)}.$$

Proof. Proof is similar to that of theorem 3.4, by using lemma (3.3). □

4. Conclusion

In this paper, we derived telescoping series for k-Fibonacci and k-Lucas sequences and proved their relationships with k-Fibonacci and k-Lucas sequences.

References

[1] Falcon, S., Plaza, A.,(2007): *On the k-Fibonacci numbers*, Chaos, Solitons and Fractals, 05(32), 1615 – 1624.

[2] Falcon, S., Plaza, A.,(2007): *The k-Fibonacci sequence and the Pascal 2 triangle*, Chaos, Solitons and Fractals, 33(1), 38 – 49.

[3] Falcon, S., Plaza, A.,(2008): *The k-Fibonacci hyperbolic functions*, Chaos, Solitons and Fractals Vol. 38(2), 409 – 20.

[4] Filipponi, P., Horadam, A. F., (1988):*A Matrix Approach to Certain Identities*, The Fibonacci Quarterly, 26(2), 115 – 126.

[5] Godase, A. D., Dhakne, M. B., (2014):*On the properties of k-Fibonacci and k-Lucas numbers*, International Journal of Advances in Applied Mathematics and Mechanics, 02(1), 100 – 106.

[6] Godase, A. D., Dhakne, M. B., (2013): *Fundamental Properties of Multiplicative Coupled Fibonacci Sequences of Fourth Order Under Two Specific Schemes*, International Journal of Mathematical Archive , 04(6), 74 – 81.

[7] Godase, A. D., Dhakne, M. B., (2013): *Recurrent Formulas of The Generalized Fibonacci Sequences of Fifth Order*, "International Journal of Mathematical Archive" , 04(6), 61 – 67.

[8] Godase, A. D., Dhakne, M. B., (2015): *Summation Identities for k-Fibonacci and k-Lucas Numbers using Matrix Methods*, International Journal of Mathematics and Scientific Computing, 05(2), 75 – 78.

[9] Godase, A. D., Dhakne, M. B., (2015): *Determinantal Identities for k-Lucas Sequence*, Journal of New Theory, 12, 01 –07.

[10] Gould, H. W., (1981) *A History of the Fibonacci q-Matrix and a Higher-Dimensional Problem*, The Fibonacci Quarterly , 19(3), 250 – 57.

[11] Horadam, A. F., (1965): *Basic Properties of a Certain Generalized Sequence of Numbers*, The Fibonacci Quarterly, 03(3), 61 – 76.

[12] Horadam, A. F., (1997): *Jacobstal Representation of Polynomials*, The Fibonacci Quarterly, 35(2), 137 – 148.

[13] Koshy, T., (2001): *Fibonacci and Lucas numbers with applications*, Wiley - Intersection Publication.

[14] Sloane, N. J., (2006): The on-line encyclopedia of integer sequences.

[15] Strang, G., (2003): *Introduction to Linear Algebra*, Wellesley-Cambridge:Wellesley, M. A. Publication.

[16] Vajda, S., (1989): *Fibonacci and Lucas numbers and the Golden Section: Theory and applications*, Chichester: Ellis Horwood,.

[17] Waddill, M. E., (1974): *Matrices and Generalized Fibonacci Sequences*, The Fibonacci Quarterly, 55(12), 381 – 386.

Department of Mathematics,, V. P. College Vaijapur,, Aurangabad(MH),, India.

E-mail address: `vpmaths@datamail.in`